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Analysis of Youden Square Design with Two Missing Observations Belonging To the Same Treatment

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ABSTRACT
Two missing observations can occur in a Youden Square Design in eight mutually exclusive ways. In the present study, the author has tried to discuss the case of two missing observations belonging to the same treatment. Estimates of the missing observations and variances of the various elementary treatment contrasts have been obtained by using Bartlett's covariate analysis.

KEYWORDS: Average variance, Adjusted treatment total, Bartlett's covariate analysis, Bias.

1. INTRODUCTION

In a Youden Square Design, n = vr experimental units are arranged in v rows, r columns and v-treatments are allocated at random to these experimental units subject to the condition that each treatment occurs once in each column and each pair of treatments occurs together in λ rows. A necessary and sufficient condition for this is that a B. I. B. Design with parameters v, b = v, r, k = r, and λ exist.

The case of one missing observation was discussed by Kshirsagar and Mckee [1](1982) pointed out that the estimate of the missing observation and variances of the various elementary treatment contrasts obtained by them seem to be incorrect. Kaushik [2](2010) discussed the case of one missing observation in a Youden Square Design in details. Later on, Kaushik A. K. and Shiv Kumar [4](2010), Kaushik A. K. and Ram Kishan[5] (2011), and Kaushik A. K. [6](2012) discussed the case of two missing observations in Youden Square Design in some special cases.

2. MATERIAL AND METHODS

This includes two sections. In section 1, the covariate analysis with two concomitant variables is presented in brief. The detailed covariate analysis pertaining to the present discussion has been discussed by Kaushik and Ram Kishan (2011). The subject matter discussed in this section is not entirely new but its presentation is new. It provides the relevant information and forms the basis of the present study. Section 2 deals with the subject matter under study.

Section 1

Covariate Analysis: The ANCOVA Model with two concomitant variables X_1 and X_2 is given below

$$Y_u = \mu + \gamma_k + \delta_j + \cdots + \rho_m + t_i + X_{1u}\beta_1 + X_{2u}\beta_2 + e_u \qquad (2.1a)$$

Its corresponding matrix model is

$$Y = Z\pi + At + X_1\beta_1 + X_2\beta_2 + e \qquad (2.1b)$$

with usual standard notations. The error sum of square will be

$$\text{E.S.S.} = \text{min. of } (Y - Z\pi - At - X_1\beta_1 - X_2\beta_2)'(Y - Z\pi - At - X_1\beta_1 - X_2\beta_2) \quad (2.2)$$

with respect to $\pi, t, \beta_1,$ and β_2 only. We get the least square estimates as below:

$$\hat{\pi} = (Z'Z)^{-1}(Z'Y - Z'\hat{A}t - Z'X_1\hat{\beta}_1 - Z'X_2\hat{\beta}_2) \quad (2.3)$$

$$\hat{t} = \bar{C}(Q_{(y)} - Q_{(X_1)}\hat{\beta}_1 - Q_{(X_2)}\hat{\beta}_2) \qquad (2.4)$$

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \begin{bmatrix}E_{X_1X_1} & E_{X_1X_2}\\ E_{X_2X_1} & E_{X_2X_2}\end{bmatrix}^{-1}\begin{bmatrix}E_{X_1Y}\\ E_{X_2Y}\end{bmatrix} \qquad (2.5)$$

Where

$$C = A'A - A'Z(Z'Z)^{-1}Z'A$$
$$Q_{(Y)} = A'Y - A'Z(Z'Z)^{-1}Z'Y$$
$$Q_{(X_1)} = A'X_1 - A'Z(Z'Z)^{-1}Z'X_1$$
$$Q_{(X_2)} = A'X_2 - A'Z(Z'Z)^{-1}Z'X_2$$
$$E_{X_1X_1} = X_1'X_1 - X_1'Z(Z'Z)^{-1}Z'X_1 - Q'_{(X_1)}\bar{C}Q_{(X_1)}$$
$$E_{X_1X_2} = X_1'X_2 - X_1'Z(Z'Z)^{-1}Z'X_2 - Q'_{(X_1)}\bar{C}Q_{(X_2)}$$
$$E_{X_2X_1} = X_2'X_1 - X_2'Z(Z'Z)^{-1}Z'X_1 - Q'_{(X_2)}\bar{C}Q_{(X_1)}$$
$$E_{X_2X_2} = X_2'X_2 - X_2'Z(Z'Z)^{-1}Z'X_2 - Q'_{(X_2)}\bar{C}Q_{(X_2)}$$
$$E_{X_1Y} = X_1'Y - X_1'Z(Z'Z)^{-1}Z'Y - Q'_{(X_1)}\bar{C}Q_{(Y)}$$
$$E_{X_2Y} = X_2'Y - X_2'Z(Z'Z)^{-1}Z'Y - Q'_{(X_2)}\bar{C}Q_{(Y)}$$

After substituting these values in (2.2), the error sum of square will be

$$\text{E. S. S.} = Y'Y - Y'Z\hat{\pi} - Y'A\hat{t} - Y'X_1\hat{\beta}_1 - Y'X_2\hat{\beta}_2$$

$$= E_{YY} - \begin{bmatrix} E_{X_1Y} & E_{X_2Y} \end{bmatrix} \begin{bmatrix} E_{X_1X_1} & E_{X_1X_2} \\ E_{X_2X_1} & E_{X_2X_2} \end{bmatrix}^{-1} \begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \end{bmatrix} \quad (2.6)$$

with $(v-2)$ d.f. only.

Under null hypothesis

$$H_0: t_1 = t_2 = \cdots = t_v = 0$$

the model (2.1) is reduced to

$$Y = Z\pi + X_1\beta_1 + X_2\beta_2 + e \quad (2.7)$$

The new error sum of square will be

$$E_0.S.S. = \text{min. of } (Y - Z\pi - X_1\beta_1 - X_2\beta_2)'(Y - Z\pi - X_1\beta_1 - X_2\beta_2) \quad (2.8)$$

with respect to $\pi, \beta_1,$ and β_2 only. We get the new least square estimates as below :

$$\pi^* = (Z'Z)^{-1}(Z'Y - Z'X_1\beta_1^* - Z'X_2\beta_2^*) \quad (2.9)$$

$$\begin{bmatrix} \beta_1^* \\ \beta_2^* \end{bmatrix} = \begin{bmatrix} E^*_{X_1X_1} & E^*_{X_1X_2} \\ E^*_{X_2X_1} & E^*_{X_2X_2} \end{bmatrix}^{-1} \begin{bmatrix} E^*_{X_1Y} \\ E^*_{X_2Y} \end{bmatrix} \quad (2.10)$$

Where

$$E^*_{X_1X_1} = X_1'X_1 - X_1'Z(Z'Z)^{-1}Z'X_1$$
$$E^*_{X_1X_2} = X_1'X_2 - X_1'Z(Z'Z)^{-1}Z'X_2$$
$$E^*_{X_2X_1} = X_2'X_1 - X_2'Z(Z'Z)^{-1}Z'X_1$$
$$E^*_{X_2X_2} = X_2'X_2 - X_2'Z(Z'Z)^{-1}Z'X_2$$
$$E^*_{X_1Y} = X_1'Y - X_1'Z(Z'Z)^{-1}Z'Y$$
$$E^*_{X_2Y} = X_2'Y - X_2'Z(Z'Z)^{-1}Z'Y$$

After substituting these values in (2.8), the error sum of square will be

$$E_0.S.S = Y'Y - Y'Z\pi^* - Y'X_1\beta_1^* - Y'X_2\beta_2^*$$

$$= E^*_{YY} - \begin{bmatrix} E^*_{X_1Y} & E^*_{X_2Y} \end{bmatrix} \begin{bmatrix} E^*_{X_1X_1} & E^*_{X_1X_2} \\ E^*_{X_2X_1} & E^*_{X_2X_2} \end{bmatrix}^{-1} \begin{bmatrix} E^*_{X_1Y} \\ E^*_{X_2Y} \end{bmatrix} \quad (2.11)$$

with $(v + v - 3)$ d.f. only.

Treatment sum of square will be obtained by

$$\text{Treatment S. S.} = E_0.S.S - E.S.S \quad (2.12)$$

with $(v-1)$ d.f. only. The variance covariance matrix will be

$$V(\hat{t}) = \bar{C}\sigma^2 + M\emptyset^{-1}M'\sigma^2 \quad (2.13)$$

Where

$$M' = \begin{bmatrix} \hat{t}_{1(X_1)} & \hat{t}_{2(X_1)} & \cdots & \hat{t}_{v(X_1)} \\ \hat{t}_{1(X_2)} & \hat{t}_{2(X_2)} & \cdots & \hat{t}_{v(X_2)} \end{bmatrix} \qquad \emptyset = \begin{bmatrix} E_{X_1X_1} & E_{X_1X_2} \\ E_{X_2X_1} & E_{X_2X_2} \end{bmatrix}$$

$$V(\hat{t}_i - \hat{t}_j) = 2\tau\sigma^2 + \begin{bmatrix} d_1 & d_2 \end{bmatrix} \emptyset^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \sigma^2 \quad (2.14)$$

Where

$d_1 = \{\hat{t}_{i(X_1)} - \hat{t}_{j(X_1)}\}$ and $d_2 = \{\hat{t}_{i(X_2)} - \hat{t}_{j(X_2)}\}$

$$\text{Average Variance} = 2a\sigma^2 + \frac{2}{(v-1)} \text{tr.} M\emptyset^{-1}M'\sigma^2 \quad (2.15)$$

Further discussion on this topic is not relevant to the present study and hence not been presented.

Section 2

Without loss of any generality, we may assume that the first k – treatments have been allotted to the first row and the first λ – treatments and (k + 1)th , (k + 2)th, ….., (2k – λ)th treatments have been allotted to the second row. Thus, both the rows have first λ treatments in common. We assume that the two missing observations belong to the first treatment in first row, first column and the first treatment in second row and second column respectively. The appropriate model for the analysis of such data is

$$Y = E\mu + At + D\gamma + F\delta + \beta_1X_1 + \beta_2X_2 + e \quad (2.16)$$

with usual notations. The covariate X_1 will assume the value ‘1’ in the first missing cell in first row and ‘0’ elsewhere while the covariate X_2 will assume the value ‘1’ in the second missing cell in second row and ‘0’ elsewhere. Now using the covariate analysis, the estimates of the missing observations are obtained as below :

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = -\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = -\emptyset^{-1} \begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \end{bmatrix}$$

$$= -\begin{bmatrix} E_{X_1X_1} & E_{X_1X_2} \\ E_{X_2X_1} & E_{X_2X_2} \end{bmatrix}^{-1} \begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \end{bmatrix}$$

Where

$$\emptyset^{-1} = \frac{\lambda v k}{k(k-2)} \begin{bmatrix} (k-1) & -1 \\ -1 & (k-1) \end{bmatrix}^{-1} = \frac{\lambda v}{k(k-2)^2} \begin{bmatrix} (k-1) & 1 \\ 1 & (k-1) \end{bmatrix} \quad (2.17)$$

and

$$\begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \end{bmatrix} = -\frac{1}{\lambda vk} \begin{bmatrix} \lambda vR_1 + \lambda kC_1 + k(kQ_1 - Q_1') - \lambda G \\ \lambda vR_2 + \lambda kC_2 + k(kQ_1 - Q_2') - \lambda G \end{bmatrix}$$

Hence

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \frac{1}{k^2(k-2)^2} \begin{bmatrix} (k-1) & 1 \\ 1 & (k-1) \end{bmatrix} \begin{bmatrix} \lambda vR_1 + \lambda kC_1 + k(kQ_1 - Q_1') - \lambda G \\ \lambda vR_2 + \lambda kC_2 + k(kQ_1 - Q_2') - \lambda G \end{bmatrix} \quad (2.18)$$

Where R_1 and R_2 are the respective totals of all the known cell observations of first and second row, C_1 and C_2 are the respective totals of all the known cell observations of first and second column, and G is the total of all the known cell observations in the experiment. Q_1 is the adjusted treatment total of first treatment.

$Q_1' = Q_1 + Q_2 + \cdots + Q_k$ = Total of all the adjusted treatment totals in the first row.

$Q_2' = Q_1 + Q_2 + \cdots + Q_\lambda + Q_{k+1} + Q_{k+2} + \cdots + Q_{2k-\lambda}$ = Total of all the adjusted treatment totals in the second row.

The error sum of square will be

$$\text{E. S. S.} = \left(\hat{Y}_1^2 + \hat{Y}_2^2 + \textstyle\sum_a Y_a^2\right) - \frac{1}{k}\left\{(R_1 + \hat{Y}_1)^2 + (R_2 + \hat{Y}_2)^2 + \textstyle\sum_{j=3}^{b} R_j^2\right\} - \frac{1}{v}\left\{(C_1 + \hat{Y}_1)^2 + (C_2 + \hat{Y}_2)^2 + \textstyle\sum_{l=3}^{k} C_l^2\right\} - \frac{k}{\lambda v}\textstyle\sum Q_i^2 + \frac{(G + \hat{Y}_1 + \hat{Y}_2)^2}{vk} \quad (2.19)$$

with {(v – 1)(k – 2) – 2} d.f. only.

Under null hypothesis

$$H_0: t_1 = t_2 = \cdots = t_v = 0$$

the model (2.16) is reduced to

$$Y = E\mu + D\gamma + F\delta + \beta_1 X_1 + \beta_2 X_2 + e$$

and we can obtain the new estimates of the missing observations as below :

$$\begin{bmatrix} Y_1^* \\ Y_2^* \end{bmatrix} = -\begin{bmatrix} \beta_1^* \\ \beta_2^* \end{bmatrix} = -\begin{bmatrix} E^*_{X_1X_1} & E^*_{X_1X_2} \\ E^*_{X_2X_1} & E^*_{X_2X_2} \end{bmatrix}^{-1} \begin{bmatrix} E^*_{X_1Y} \\ E^*_{X_2Y} \end{bmatrix} = \frac{1}{(k-1)^2(v-1)^2-1}\begin{bmatrix} (k-1)(v-1) & -1 \\ -1 & (k-1)(v-1) \end{bmatrix}\begin{bmatrix} vR_1 + kC_1 - G \\ vR_2 + kC_2 - G \end{bmatrix} \quad (2.21)$$

The error sum of square under the model (2.19) will be

$$E_0.\ S.\ S. = (Y_1^{*2} + Y_2^{*2} + \textstyle\sum_a Y_a^2) - \frac{1}{k}\{(R_1 + Y_1^*)^2 + (R_2 + Y_2^*)^2 + \sum_{j=3}^{b} R_j^2\} - \frac{1}{v}\{(C_1 + Y_1^*)^2 + (C_2 + Y_2^*)^2 + \sum_{l=3}^{k} C_l^2\} + \frac{(G + Y_1^* + Y_2^*)^2}{vk} \quad (2.22)$$

with {(v – 1)(k – 1) – 2} d.f. only.

Treatment sum of square will be obtained by

Treatment S. S. = E_0.S.S – E.S.S (2.23)

with $(v-1)$ d.f. only. The variance covariance matrix will be

$$V(\hat{t}) = \frac{k\sigma^2}{\lambda v} I_v + M\emptyset^{-1}M'\sigma^2 \quad (2.24)$$

Where

$$M' = \frac{1}{\lambda v}\begin{bmatrix} k-1 & -1 & \cdots & -1 & -1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ k-1 & -1 & \cdots & -1 & 0 & \cdots & -1 & \cdots & -1 & 0 & \cdots & 0 \end{bmatrix} \quad (2.25)$$

The variances of various elementary treatment contrasts are given below:

$$V(\hat{t}_1 - \hat{t}_u) = \frac{2k\sigma^2}{\lambda v} + \frac{2k^2\sigma^2}{\lambda v(k-2)^2} \quad (2.26)$$

(u = 2, 3, … , λ)

$$V(\hat{t}_1 - \hat{t}_w) = V(\hat{t}_1 - \hat{t}_h) = \frac{2k\sigma^2}{\lambda v} + \frac{(k-1)(2k^2+1)\sigma^2}{\lambda vk(k-2)^2} \quad (2.27)$$

(w = λ+1, λ+2, … , k ; h = k+1, k+2, …, 2k – λ)

$$V(\hat{t}_1 - \hat{t}_g) = \frac{2k\sigma^2}{\lambda v} + \frac{2(k-1)^2\sigma^2}{\lambda v(k-2)^2} \quad (2.28)$$

(g = 2k – λ +1, 2k – λ +2, …, v)

$$V(\hat{t}_u - \hat{t}_w) = V(\hat{t}_u - \hat{t}_h) = V(\hat{t}_w - \hat{t}_g) = V(\hat{t}_h - \hat{t}_g) = \frac{2k\sigma^2}{\lambda v} + \frac{(k-1)\sigma^2}{\lambda vk(k-2)^2} \quad (2.29)$$

$$V(\hat{t}_u - \hat{t}_g) = \frac{2k\sigma^2}{\lambda v} + \frac{2\sigma^2}{\lambda v(k-2)^2} \quad (2.30)$$

$$V(\hat{t}_w - \hat{t}_h) = \frac{2k\sigma^2}{\lambda v} + \frac{2\sigma^2}{\lambda vk(k-2)} \quad (2.31)$$

$$V(\hat{t}_u - \hat{t}_{u'}) = V(\hat{t}_w - \hat{t}_{w'}) = V(\hat{t}_g - \hat{t}_{g'}) = V(\hat{t}_h - \hat{t}_{h'}) = \frac{2k\sigma^2}{\lambda v} \quad (2.32)$$

This is to be noted that the values of various variances of the elementary treatment contrasts get increased when missing observations occur.

$$\text{Average Variance} = \frac{2k\sigma^2}{\lambda v} + \frac{4\{(k-1)^2(k+1)+(\lambda-1)\}\sigma^2}{\lambda vk(v-1)(k-2)^2} \quad (2.33)$$

$$\text{Relative Efficiency} = \frac{k^2(v-1)(k-2)^2}{k^2(v-1)(k-2)^2+2\{(k-1)^2(k+1)+(\lambda-1)\}} \quad (2.34)$$

Relative Loss in Efficiency = 1 – R.E =

$$\frac{2(k-1)^2(k+1)+(\lambda-1)}{k^2(v-1)(k-2)^2+2\{(k-1)^2(k+1)+(\lambda-1)\}} \quad (2.35)$$

$$\text{Bias} = \frac{(v-1)(k-1)}{vk}\{(\hat{Y}_1 - Y_1^*)^2 + (\hat{Y}_2 - Y_2^*)^2\} + \frac{2}{vk}(\hat{Y}_1 - Y_1^*)(\hat{Y}_2 - Y_2^*) \quad (2.36)$$

Illustration: Consider the data obtained from a Youden Square Design with parameters v = b = 5, r = k = 4, λ = 3. The two missing observations belong to first treatment A from the first two rows.

Rows	Columns			
	I	II	III	IV
I	A = –	B = 20	C = 23	D = 25
II	B = 16	A = –	E = 27	C = 29
III	C = 14	D = 19	A = 12	E = 20
IV	D = 18	E = 18	B = 15	A = 13
V	E = 17	C = 16	D = 24	B = 20

By using (2.18), and (2.19), we get

$\hat{Y}_1 = 11.25$, $\hat{Y}_2 = 16.75$, $E.S.S. = 27.1333$ with 6 df only.

By using (2.21), and (2.22), we get

$Y_1^* = 19.1748$, $Y_2^* = 23.9021$, $E_0.S.S. = 165.2776$ with 10 df only.

By using (2.23), we get

Treatment S. S. = E_0.S.S. – E.S.S. = 138.1443 with 4 df only.

The variances of elementary treatment contrast are obtained as

$$V(\hat{t}_A - \hat{t}_B) = V(\hat{t}_A - \hat{t}_C) = \frac{8\sigma^2}{15} + \frac{8\sigma^2}{15}$$

$$V(\hat{t}_A - \hat{t}_D) = V(\hat{t}_A - \hat{t}_E) = \frac{8\sigma^2}{15} + \frac{33\sigma^2}{80}$$

$$V(\hat{t}_B - \hat{t}_D) = V(\hat{t}_C - \hat{t}_D) = V(\hat{t}_B - \hat{t}_E) = V(\hat{t}_C - \hat{t}_E) = \frac{8\sigma^2}{15} + \frac{\sigma^2}{80}$$

$$V(\hat{t}_D - \hat{t}_E) = \frac{8\sigma^2}{15} + \frac{\sigma^2}{30}$$

$$V(\hat{t}_B - \hat{t}_C) = \frac{8\sigma^2}{15}$$

Average Variance = $\frac{8\sigma^2}{15} + \frac{47\sigma^2}{240}$

Relative Efficiency = $\frac{64}{77}$

Relative Loss in Efficiency = $\frac{13}{77}$

CONCLUSION

The estimates of missing observations ar

$Y_1^*(A) = 11.25$, $Y_2^*(A) = 16.75$,

By using normal procedure, the treatments can be tested for homogeneity. Various contrasts of treatment effects are obtained in the example.

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