International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 08 Issue 05 May 2020, Page no. -2046-2052

Index Copernicus ICV: 57.55

DOI: 10.33826/ijmcr/v8i5.01

The Solution for First Order Differential Inequalities in Banach Algebra

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ARTICLE INFO	ABSTRACT
Published Online: 01 May 2020 Corresponding Author: **N. S. Pimple**	The present paper reveals the existence theorem for the first order functional differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Caratheodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

KEYWORDS: Functional differential equation, Banach algebra, Lipschitz conditions, Caratheodory conditions, Extremal solutions.

1. INTRODUCTION

In this paper we study the solution of first order differential inequalities in Banach algebra using some mixed generalized Lipschitz and Caratheodory conditions. Being the most striking field of research, the functional differential equations come forth rigorously. There are Hale[10], Henderson[11] and the references for consideration.. But the study of functional differential equations in Banach algebra is very rare in the literature. Very recently the study in favor of this line has been initiated via fixed point theorems. see Dhage and Regan[6] and the references therein.

1.1. Statement of the Problem

Let R denote the real line and let $I_0=[r,0]\ and\ I=[0,a]$ be two closed and bounded intervals in R. Let $J=[-r,0]\bigcup[0,a]$, then J is a closed and bounded interval in R.Let C denote the Banach space of all continuous real valued functions φ on I_0 with the supremum norm $\|\cdot\|_C$ defined by

$$\|\varphi\|_C=\sup_{t\in I_0}|\varphi(t)|.$$

Clearly C is a Banach algebra with respect to this norm and the multiplication "." defined by

$$(xy)(t)=x(t)y(t),\quad t\in I_0.$$

Consider the first order functional differential equation (FOFDE)

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{x(t)}{f(t,x(\alpha(t)))}\right)&=g(t,x_{\alpha t})\ \ a.e.\ \ t\in I\\ x(t)&=\varphi(t)\,,\,t\in I_0,\end{aligned}\right\}\qquad(1.1)$$

Where $f:I\times R\to R-\{0\}$ is continuous, $g:I\times C\to R$, $\alpha:I\to I$ and the function $x_{\alpha t}(\theta):I_0\to C$ is defined by

$$x_{\alpha t}(\theta)=x(\alpha t+\theta)\ \ for\ all\ \theta\in I_0.$$

By a solution of FOFDE 1.1) we mean a function $x\in C(J,R)\bigcap AC(I,R)\bigcap C(I_0,R)$ that satisfies the equation in (1.1), where $AC(I,R)$ is the space of all absolutely continuous real - valued functions on J.

The FOFDE (1.1) is new way of thinking to the literature and the study of this problem will definitely contribute a lot to the area of functional differential equations.

2. AUXILIARY RESULTS

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A:X\to X$ is called D-Lipschitzian if there exists a continuous nondecreasing function $\psi:R^+\to R^+$ satisfying

$$\|Ax-Ay\|\le\psi(\|x-y\|)\qquad(2.1)$$

for all $x,y\in X\ \ with\ \psi(0)=0$. In the special case when $\psi(r)=\eta r\ (\eta>0)$, A is called a Lipchitzian with a Lipschitz constant η. In particular, if $\eta<1$,A is called a contraction with a contraction constant η. Further, if

$\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X. Sometimes we call the function ψ a D-function for convenience.

An operator $T : X \to X$ is called compact if $\overline{T(X)}$ is a compact subset of X. Similarly, $T : X \to X$ is called totally bounded if T maps a bounded subset of X into the relatively compact subset of X. Finally, $T : X \to X$ is called completely continuous operator if it is continuous and totally bounded operator on X. It is very obvious that every compact operator is completely bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage[5] is embodied in the following theorem.

Theorem 2.1 ([5]) Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying

(a) A is a D-Lipschitzian with a D-function ψ,

(b) B is compact and continuous, and

(c) $M\psi(r) < r$ *whenever* $r > 0$, where

$$M = \|\beta(X)\| = \sup\{\|\beta(X)\| : x \in X\}$$

Then either

(i) The equation $\lambda A\left(\frac{x}{\lambda}\right)Bx = x$ has a solution for $\lambda = 1$, or

(ii) Theset $\varepsilon = \{u \in X : \lambda A(u/\lambda)\mathrm{Bx} = \mathrm{u}, 0 < \lambda < 1\}$ is unbounded.

We are aware of that theorem 2.1which is beneficial for proving the existence theorems for the integral equations of mixed type. See Bellale [3] and the references therein. The method is commonly known as *priori bound method* for the nonlinear equations. See, for example, Dugunji and Granas [7],Zeidler[12] and the references therein.

An interesting corollary to Theorem 2.1 in its applicable form is

Corollary 2.1 Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying

(a) A is Lipschitzian with a Lipschitz constant η,

(b) B is compact and continuous, and

(c)

$$\eta M < 1, \ where \ M = \|\beta(\mathrm{X})\| := \sup\{\|\beta(\mathrm{X})\| : x \in X\}$$

then either

(i) The equation $\lambda A\left(\frac{x}{\lambda}\right)Bx = x$ has a solution for $\lambda = 1$, or

(ii) The set $\varepsilon = \{u \in X : \lambda A(u/\lambda)\mathrm{Bx} = \mathrm{u}, 0 < \lambda < 1\}$ is unbounded.

3. EXISTENCE THEORY

Let M(J,R) and B(J,R) respectively denote the spaces of measurable and bounded real-valued functions on J.

We shall seek the existence of a solution of FOFDE (1.1) in the space C(J,R), of all absolutely continuous real – valued functions on J. Define a norm $\|\cdot\|$ in C(J,R) by

$$\|x\| = \sup_{t \in J} |x(t)|$$

Clearly C(J,R) becomes a Banach algebra with this norm. Note that $C(\mathrm{J,R}) \subset \mathrm{AC(J,R)}$.

We take assistance of the following definition in the sequel.

Definition 3.1([7]) A mapping $\beta : I \times C \to R$ is said to satisfy the condition of $L^1_{X^-}$ Caratheodory or simply said to be $L^1_{X^-}$ Caratheodory if

(i) $t \to \beta(\mathrm{t, x})$ is measurable for each $x \in C$

(ii) $x \to \beta(t, x)$ is a continuous almost everywhere for $t \in I$, and

(iii) there exists a function $h \in L^1(I, R)$ such that

$$|\beta(t, x)| \le h(t), \ a.e. \ t \in I$$

for all $x \in R$.

For convenience, the function h is referred to as a bound function of β

We will need the following hypothesis in the sequel.

(H_1): The function $f : I \times R \to R$ is continuous and there exists a function $k \in B(I, R)$ such that $k(t) > 0, \ a.e. \, t \in I$ and

$$|f(t,x) - f(t,y)| \le k(t)|x - y|, \ a.e. \ t \in I$$

for all $x, y \in R$.

(H_2) : $f(0, \phi(0)) = 1$.

(H_3) : The function $g(t, x)$ is $L^1_{X^-}$ **Caratheodory** with the bound function h.

(H_4) : There exists a continuous and nondecreasing function $\Im : [0, \infty) \to (0, \infty)$ and a function $\rho \in L^1(\mathrm{I, R})$ such that $\rho(\mathrm{t}) > 0, \ \mathrm{a.e.} \ \mathrm{t} \in \mathrm{I}$

and $$|g(t,x)| \le \rho(t)\Im\left(\|x\|_C\right), \ a.e. \ t \in I,$$

for all $x \in C$.

Theorem 3.1 Assume that the hypothesis (H_1)-(H_4) hold. Suppose that

$$\int_{C_1}^{\infty} \frac{ds}{\Im(s)} > C_2 \|\rho\|_{L^1} \qquad (3.1)$$

Where,

$$C_1 = \frac{F\|\phi\|_C}{1 - \|k\|\left(\|\phi\|_C + \|h\|_{L^1}\right)}, \quad C_2 = \frac{1}{1 - \|k\|\left(\|\phi\|_C + \|h\|_{L^1}\right)}, \quad \|k\|\left(\|\phi\|_C + \|h\|_{L^1}\right) < 1$$

$F = \max_{t\in J}|f(t,0)|$, *and* $\|k\| = \max_{t\in J}|k(t)|$. Then the FOFDE (1.1) has a solution on J.

Proof: Now the FOFDE (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = [f(t, x(\alpha(t)))]\left(\varphi(0) + \int_0^t g(s, x_{\alpha s})ds\right), \quad if\ t \in I \tag{3.2}$$

and $\quad x(t) = \varphi(t), \quad if \quad t \in I_0 \quad$ (3.3)

Define the two mappings A and B on C(J,R) by

$$Ax(t) = \begin{cases} f(t, x(t)), & if\ t \in I, \\ 1 & ,\ if\ t \in I_0 \end{cases} \tag{3.4}$$

and

$$Bx(t) = \begin{cases} \varphi(0) + \int_0^t g(s, x_{\alpha s})ds, & if\ t \in I, \\ \varphi(t) & ,\ if\ t \in I_0 \end{cases} \tag{3.5}$$

Obviously A and B define the operators $A, B : C(J,R) \to C(J,R)$. then the FOFDE (1.1) is equivalent to the operator equation

$$x(t) = Ax(t)Bx(t), \quad t \in J \tag{3.6}$$

Our motive is to show the operators A and B satisfy all the hypothesis of corollary 2.1

We initially show that A is Lipschitzian on C(J,R). Let $x, y \in C(J,R)$. then by (H_1),

$$|Ax(t) - Bx(t)| = |f(t, x(\alpha(t)) - f(t, y(\alpha(t))|$$

$$\le k(t)|x(\alpha(t)) - y(\alpha(t))|$$

$$\le k(t)\|x - y\|$$

for all $t \in J$ .Taking the supremum over t, we have

$$\sup_{t\in J}|Ax(t) - Bx(t)| \le \sup_{t\in J}\left[k(t)\|x - y\|\right]$$

thus we obtain

$$\|Ax - Ay\| \le \|k\|\|x - y\|$$

for all $x, y \in C(J,R)$. So A is a Lipschitzian on C(J,R) with a Lipschitz constant $\|k\|$.

Next we will project that B is completely continuous on C(J,R).Using the standard arguments as in Granas et al.[11],it is shown that B is a continuous operator on C(J,R).

Let S be a bounded set in C(J,R). We shall show that B (C(J,R)) is a uniformly bounded and equicontinuous set in C(J,R).Since g(t, x_t) is $L^1{}_{X^-}$ Caratheodory ,

we have

$$|Bx(t)| = \left|\varphi(t) + \int_0^t g(s, x_{\alpha s})ds\right|$$

$$\le |\varphi(t)| + \left|\int_0^t g(s, x_{\alpha s})ds\right|$$

$$\le \|\varphi\|_C + \int_0^t |g(s, x_{\alpha s})|ds$$

$$\therefore |Bx(t)| \le \|\varphi\|_C + \|h\|_{L^1}$$

Taking the supremum over t, we obtain $\|Bx\| \le M \quad for\ \ all\ \ x \in S$,where

$$M = \|\phi\|_C + \|h\|_{L^1}.$$

This shows that B(C(J,R)) is uniformly bounded set in C(J,R).

Now we show that B(C(J,R)) is an equicontinuous set. $t_1, t_2 \in I$. Then for any $x \in C(J,R)$ we have by (3.5),

$$|Bx(t_1) - Bx(t_2)| = \left|\varphi(0) + \int_0^{t_1} g(s, x_{\alpha s})ds - \varphi(0) - \int_0^{t_2} g(s, x_{\alpha s})ds\right|$$

$$= \left|\int_0^{t_1} g(s, x_{\alpha s})ds - \int_0^{t_2} g(s, x_{\alpha s})ds\right|$$

$$= \left|\int_0^{t_1} g(s, x_{\alpha s})ds + \int_{t_2}^{0} g(s, x_{\alpha s})ds\right|$$

$$= \left|\int_{t_2}^{t_1} g(s, x_{\alpha s})ds\right|$$

$$\le \int_{t_2}^{t_1} |g(s, x_{\alpha s})|ds$$

$$\le \int_{t_2}^{t_1} |h(s)|ds \le |p(t_1) - p(t_2)|$$

$$where \quad p(t) = \int_0^t h(s)ds.$$

Therefore

$$|Bx(t_1) - Bx(t_2)| \to 0\ as\ t_2 \to t_1$$

Again let $t_2 \in I_0, t_1 \in I$, then we obtain

$$|Bx(t_1)-Bx(t_2)| = \left|\varphi(0)+\int_0^{t_1} g(s,x_{\alpha s})ds-\varphi(t_2)\right|$$
$$= \left|\varphi(0)-\varphi(t_2)+\int_0^{t_2} g(s,x_{\alpha s})ds+\int_{t_2}^{t_1} g(s,x_{\alpha s})ds\right|$$
$$\le |\varphi(t_2)-\varphi(0)|+\left|\int_0^{t_2} g(s,x_{\alpha s})ds\right|+\left|\int_{t_2}^{t_1} g(s,x_{\alpha s})ds\right|$$
$$\le |\varphi(t_2)-\varphi(0)|+\int_0^{t_2} |g(s,x_{\alpha s})|ds+\int_{t_2}^{t_1} |g(s,x_{\alpha s})|ds$$
$$\le |\varphi(t_2)-\varphi(0)|+|p(t_2)|+|p(t_1)-p(t_2)|$$

where the function p is defined above. Similarly, if $t_1,t_2 \in I_0$ then we get

$$|Bx(t_1)-Bx(t_2)| = |\varphi(t_1)-\varphi(t_2)|.$$

therefore in all above three cases

$$|Bx(t_1)-Bx(t_2)| \to 0 \; as \; t_2 \to t_1 \; \forall t_1,t_2 \in J.$$

Hence B(C(J,R)) is an equicontinuous set and consequently B(C(J,R)) is relatively compact by Arzela-Ascoli theorem. As a result B is a compact and continuous operator on C(J,R)).Thus all the conditions of theorem 2.1 are to the requirements and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds.

We express here that the conclusion (ii) is not possible.

Let $x \in X$ be any solution to FOFDE (1.1). Then we have, for any $\lambda \in (0,1)$,

$$x(t) = \lambda A\left(\frac{x}{\lambda}\right)(t)\cdot Bx(t)$$
$$= \begin{cases} \lambda\left[f(t),\dfrac{x(\alpha(t))}{\lambda}\right]\cdot\left[\varphi(0)+\displaystyle\int_0^t g(s,x_{\alpha s})ds\right], & t \in I \\ \lambda(1)\cdot\varphi(t) & , \; t \in I_0 \end{cases}$$

for $t \in J$. Thus if $t \in I_0$, then

$$|x(t)| \le |\lambda||\varphi(t)|$$
$$\le |\lambda|\sup_{t\in I_0}|\varphi(t)|$$
$$|x(t)| \le \|\varphi\|_{C.}$$

Again if $t \in I$, then we have

$$|x(t)| = \left|\lambda\left(f(s,\frac{x(\alpha(t))}{\lambda}\right)\cdot\left(\varphi(0)+\int_0^t g(s,x_{\alpha s})ds\right)\right|$$
$$\le \lambda\left|f(s,\frac{x(\alpha(t))}{\lambda}\right|\left\{|\varphi(0)|+\left|\int_0^t g(s,x_{\alpha s})ds\right|\right\}$$
$$\le \lambda\left\{\left|f(s,\frac{x(\alpha(t))}{\lambda}-f(t,0)\right|+|f(t,0)|\right\}\left\{\|\varphi\|_C+\int_0^t g(s,x_{\alpha s})ds\right\}$$
$$\le \lambda\left[k(t)\left|\frac{x(\alpha(t))}{\lambda}-0\right|+\frac{F}{\lambda}\right]\left\{\|\varphi\|_C+\int_0^t g(s,x_{\alpha s})ds\right\}$$
$$\le \left[k(t)|x(\alpha(t))|+F\right]\left\{\|\varphi\|_C+\int_0^t g(s,x_{\alpha s})ds\right\}$$
$$\le k(t)|x(\alpha(t))|\left\{\|\varphi\|_C+\int_0^t g(s,x_{\alpha s})ds\right\}+F\left\{\|\varphi\|_C+\int_0^t g(s,x_{\alpha s})ds\right\}$$
$$\le \|k\|\|x_{\alpha t}\|\left[\|\varphi\|_C+\|h\|_{L^1}\right]+F\|\varphi\|_C+F\int_0^t \rho(s)\cdot\Im\left(\|x_{\alpha s}\|_C\right)ds \tag{3.7}$$

put $u(t) = \sup_{s\in[-r,t]}|x(s)|, \quad for \; t \in J.$

Then we have

$$|x(t)| \le u(t) \quad \forall t \in J \; and \; \|x_{\alpha t}\|_C \le u(t), \; \forall t \in I$$

and so, there is a point $t^+ \in [-r,t]$ such that

$$u(t) = |x(t^+)|$$

From (3.7) it follows that

$$u(t) = |x(t^+)|$$
$$\le \|k\||x(t^+)|\left(\|\varphi\|_C+\|h\|_{L^1}\right)+F\left(\|\varphi\|_C+\int_0^{t^+}\rho(s)\Im\left(\|x_{\alpha s}\|_C\right)ds\right)$$
$$\le \|k\|u(t)\left(\|\varphi\|_C+\|h\|_{L^1}\right)+F\left(\|\varphi\|_C+\int_0^t\rho(s)\Im(u(s))\,ds\right)$$
$$\le C_1+C_2\int_0^t\rho(s)\Im(u(s))\,ds \tag{5.3.8}$$

where

$$C_1 = \frac{F\|\varphi\|_C}{1-\|k\|\left[\|\varphi\|_C+\|h\|_{L^1}\right]} \quad and \quad C_2 = \frac{1}{1-\|k\|\left[\|\varphi\|_C+\|h\|_{L^1}\right]}$$

Let

$$q(t) = C_1+C_2\int_0^t\rho(s)\cdot\Im(u(s))\,ds.$$

then $u(t) \le q(t)$ and a direct differentiation of q(t) yields

$$\left.\begin{aligned} q'(t) &= C_2\rho(t)\Im(q(t)) \\ q(0) &= C_1 \end{aligned}\right\} \tag{3.9}$$

that is

$$\int_0^t \frac{q'(s)}{\Im(q(s))}\,ds \le C_2 \int_0^t \rho(s)\,ds \le C_2 \|\rho\|_{L^1}$$

A change of variables in the above integral mentions that

$$\int_{C_1}^{q(t)} \frac{ds}{\Im(s)} \le C_2 \|\rho\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Im(s)}.$$

Now an application of mean value theorem yields that there is a constant M>0 such that

$$q(t) \le M \ \ for\ all\ t \in J.$$

This further implies that $|x(t)| \le u(t) \le q(t) \le M$,

for all $t \in J$.

Thus, the conclusion (ii) of corollary 2.1 does not hold. Therefore, the operator equation $AxBx = x$ and consequently the FOFDE (1.1) has a solution on J.

4. EXISTENCE OF EXTREMAL SOLUTIONS

A non-empty closed set K in a Banach Algebra X is called a **cone** if

(i) $K + K \subseteq K$,

(ii) $\lambda K \subseteq K\ for\ \lambda \in \mathbb{R}, \lambda \ge 0$ and

(iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X .

A cone K is called to be **positive** if

(iv) $K \circ K \subseteq K$, $where$ "$\circ$" is a multiplication composition in X.

We introduce an order relation $\le$ in K as follows:

Let $x, y \in X$, then $x \le y\ if\ and\ only\ if\ y - x \in K.$

A cone is said to be normal if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bound. The details of cones and their properties appear in Guo and Lakshmikantham [9].

We equip the space $C(J, \mathbb{R})$ with the order relation $\le$ with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \ge 0, \forall t \in J\}. \qquad (4.1)$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$. As a result of positivity of the cone K in $C(J, \mathbb{R})$ we have:

Lemma 4.1 ([14]) Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \le v_1\ and\ u_2 \le v_2$. Then $u_1 u_2 \le v_1 v_2$.

For any $a, b \in X = C(J, \mathbb{R}), a \le b$, the order interval [a, b] is a set in X given by

$$[a, b] = \{x \in X : a \le x \le b\}.$$

We make use of the following fixed point theorem, for proving the existing of extremal solutions of the FOFDE (1.1) under certain monotonicity conditions.

Theorem 4.1 ([6]) Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that

(a) A is Lipschitzian with a Lipschitz constant α,

(b) B is completely continuous,

(c) $Ax\,Bx \in [a, b]\ \ for\ each\ x \in [a, b]$, and

(d) A and B are nondecreasing.

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a maximal and a minimal positive solution in $[a, b]$, whenever $\eta M < 1$, where

$$M = \| B([a,b]) \| := \sup\{\|Bx\| : x \in [a, b]\}.$$

We need the following definitions in the sequel.

Definition 4.1 ([14]) A function $u \in C(J, \mathbb{R})$ is called a lower solution of the FOFDE (1.1) on J if

$$\frac{d}{dt}\left(\frac{u(t)}{f(t, u(a(t)))}\right) \le g(t, u_{at}),\ a.e\ t \in I$$

and

$$u(t) \le \varphi(t)\ \ for\ all\ \ t \in I_0.$$

Again a function $v \in C(J, \mathbb{R})$ is called an upper solution of the BVP (4.1) on J if

$$\frac{d}{dt}\left(\frac{v(t)}{f(t, v(a(t)))}\right) \ge g(t, v_{at}),\ a.e\ t \in I$$

and

$$v(t) \ge \varphi(t)\ \ for\ all\ \ t \in I_0.$$

Definition 4.2 ([4]) A solution x_M of the FOFDE (1.1) is said to be maximal if for any other solution x to FOFDE (1.1) one has $x(t) \le x_M(t),\ \forall t \in J$. Again a solution x_m of the FOFDE (1.1) is said to be minimal if $x_m(t) \le x(t),\ \forall t \in J$, where x is any solution of the FOFDE (1.1) on J.

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R}^+ \to \mathbb{R}^+ - \{0\},\ g : J \times C \to \mathbb{R}^+$ and $\varphi(0) \ge 0.$

(B_1) $g\ is\ L^1_X$- Caretheodory.

(B_2) The function $f(t, x)\ and\ g(t, y)$ are nondecreasing in $x\ and\ y$ almost everywhere for $t \in J$.

(B_3) The FOFDE (1.1) has a lower solution u and an upper solution v on $J\ with\ u \le v.$

Remark 4.1 Assume that (B_1) - (B_3) hold. Define a function $h : J \to \mathbb{R}^+$ by

$$h(t) = |g(t, u_{at})| + |g(t, v_{at})|,\ \forall t \in I.$$

Then h is Lebesgue integrable and

$$\left|g(t,x_{at})\right| \le h(t),\ a.e.\ t \in I, \forall x \in [u,v].$$

Theorem 4.2 Suppose that the assumptions (H_1) – (H_3) and (B_0) to (B_3) hold. Further if $\|k\|\left(\|\varphi\|_C + \|h\|_{L^1}\right) < 1$, and h is given in Remark 4.1, then FOFDE (1.1) has a minimal and a maximal positive solution on J.

Proof: Now FOFDE (1.1) is equivalent to FIE (3.2)-(3.3) on J .Let $X = C(J, \mathbb{R})$.

Define two operators A *and* B *on* X by (3.4) and (3.5) respectively. Then FIE (1.1) transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach Algebra X. Notice that (B_1) implies $A, B:[u,v] \to K.$ Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X. Now it is forwarded as in the proof of Theorem (3.1) , that A is a Lipschitzian with a Lipschitz constant $\|h\|$ and B is completely continuous operator on $[u,v]$. Again the hypothesis (B_2) implies that A *and* B are nondecreasing on $[u,v]$ .To see this, let $x, y \in [u,v]$ be such that $x \le y.$ Then by (B_2) ,

$$Ax(t) = f(t, x(a(t)) \le f(t, y(a(t)) = Ay(t)\ , \forall t \in I,$$

and

$$Ax(t)\ = 1 = Ay(t), \quad for\ all\ \ t \in I_0.$$

Similarly

$$\begin{aligned} Bx(t) &= \varphi(0) + \int_0^t g(s, x_{\alpha s})\, ds \\ &\le \varphi(0) + \int_0^t g(s, y_{\alpha s})\, ds \\ &= Ay(t),\ \ \forall t \in I, \end{aligned}$$

and $\quad Bx(t) = \varphi(t) = By(t) \quad for\ all\ \ t \in I_0.$

So A *and* B are nondecreasing operators on $[u,v]$ .Again Lemma 4.1 and hypothesis (B_3) implies that

$$\begin{aligned} u(t)\ &\le\ [f(t,u(\alpha(t)))]\left(\varphi(0) + \int_0^t g(s,u_{\alpha s})\, ds\right) \\ &\le\ [f(t,x(\alpha(t)))]\left(\varphi(0) + \int_0^t g(s,x_{\alpha s})\, ds\right) \\ &\le\ [f(t,v(\alpha(t)))]\left(\varphi(0) + \int_0^t g(s,v_{\alpha s})\, ds\right) \\ &\le\ [f(t,u(\alpha(t)))]\left(\varphi(0) + \int_0^t g(s,u_{\alpha s})\, ds\right) \\ &\le\ v(t), \end{aligned}$$

for all $t \in I$ *and* $x \in [u,v].$ As a result

$$u(t) \le Ax(t)Bx(t) \le v(t),\ \forall t \in J\ and\ x \in [u,v].$$

Hence, $AxBx \in [u,v],\ \forall x \in [u,v].$

Again

$$\begin{aligned} M\ &=\ \left\|B([u,v])\right\| \\ &=\ \sup\left\{\|Bx\| : x \in [u,v]\right\} \\ &\le\ \sup\left\{\|\varphi\|_C + \sup_{t\in J}\int_0^t \left|g(s,x_{\alpha s})\right| ds : x \in [u,v]\right\} \\ &\le\ \|\varphi\|_C + \int_0^a h(s)\, ds \\ &=\ \|\varphi\|_C + \|h\|_{L^1}. \end{aligned}$$

Since $\eta M \le \|k\|\left(\|\varphi\|_C + \|h\|_{L^1}\right) < 1,$ we apply Theorem 4.1 to the operator equation $Ax(t)Bx(t) = x$ to yield that the FOFDE (1.1) has a minimal and a maximal positive solution on J .This completes the proof.

5. References

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