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Generalized Independent Sets in a Graph

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ARTICLE INFO	ABSTRACT
Published Online: 14 May 2019Corresponding Author: **V. Swaminathan**	Independent sets in a graph can be generalized in several ways. E. Sampathkumar introduced I_k-sets as a generalization of independent sets. In this paper, star I_k-sets are defined and studied.

KEYWORDS: Independent set, I_k-set, star I_k-set.

1 Introduction

Let $G = (V, E)$ be a simple graph. E. sampathkumar [5] introduced generalized independent sets.According to him, a subset S of $V(G)$ is called a generalized independent set of G if there exists a positive integer $k, k \geq 2$ such that any k- vertices of S induced a disconnected subgraph of G. Such a set is called an I_k-set of G. The maximum cardinality of such a set is denoted by $\beta_{0k}(G)$ and is called the k-independent number of G. Several results on I_k-sets were derived in[5].In this paper, star I_k-sets are defined.The maximum cardinality of a star I_k-set denoted by β_{kst}(G) is found for some well known graphs.Also graphs are characterized whose β_{kst} - values are specified.

Definition 0.1 [5]
Let $G = (V, E)$ be a simple finite and undirected graph. Let S be a subset of $V(G)$, S is called a k- independent $(I_k - \text{set})(k \geq 2)$ of G if for any subset T of S with cardinality k, $< T >$ is disconnected.

Remark 0.1 [5]
Let $k \geq 2$. A subset S with $|S| < k$ is assumed to be an I_k - set.If $|S| = k$,then S is an I_k - set if $< S >$ is disconnected.

Remark 0.2 [5]
Any I_k - set with k or more than k- elements cannot contain a full degree vertex. Also, any I_k - set with k or more than k- elements is disconnected.

1.1 Star I_k - sets

Definition 1.1
A subset S of $V(G)$ is called a star I_k - set of G if S is an I_k - set of G and $S \subset N(v)$ for some $v \in V(G)$. That is S is dominated by v in G. S is called an I_{kst} - set of G.

Definition 1.2
The maximum cardinality of a star I_k - set of G is called the star I_k - number of G and it is denoted by $\beta_{kst}(G)$.

Remark 1.1
$\beta_{ost}(G) \leq \beta_{kst}(G) \leq \beta_{rst}(G)$,where $2 \leq k \leq r$.Clearly, $\beta_{ost}(G) \leq \beta_o(G)$, $\beta_{kst}(G) \leq \beta_{ok}(G)$

Definition 1.3 [2]
A subset S of $V(G)$ is called star independent, if S is independent and $S \subset N(v)$ for some $v \in V(G)$.The maximum cardinality of a star independent set of G is called the star independent number of G and it is denoted by $\beta_{st}(G)$.

Theorem 1.1
Let G be a simple connected graph. Then $\beta_{kst}(G) = 1$ if and only if G is either a complete graph $K_r, r \geq 2$ with $k = 2$ or K_2 with $k > 2$.

Proof:
Suppose $\beta_{kst}(G) = 1$. Then $\beta_{st}(G) \leq \beta_{kst}(G) = 1$. Therefore, $\beta_{st}(G) = 1$. Therefore, G is complete (by theorem 2.1.5) [2]
Suppose $G = K_r$.

$$1 = \beta_{kst}(G) = \begin{cases} k - 1 \, if \, r \geq k \\ r - 1 \, if \, r < k \end{cases}$$

Therefore, $k = 2$ if $r \geq k$ or $r = 2$ if $r < k$. Therefore,$\beta_{kst}(G) = 1$ implies $k = 2$ or $G = K_2$, $2 < k$, which implies G is any complete graph when $k = 2$ or G is K_2 where $r < k$
Conversely, Let G be a complete graph, when $k = 2$. $\beta_{kst}(G) = 1$. Suppose G is K_2 where $2 < k$. Then, $\beta_{kst}(G) = 1$

Corollary 1.1
Let G be a simple disconnected graph. Then $\beta_{kst}(G) = 1$ if and only if $k = 2$ and every component of G is a complete graph K_r with $r \geq 2$ or $k > 2$ and every component is a K_2.

Theorem 1.2
$\beta_{kst}(G) = 2$ if and only if for any $u \in V(G)$, every component of $N(u)$ is complete and the order r of any component is at least k where $k = 3$ or order of any component is 2 where $k > 3$.

Proof:
Case (i) Let $u \in V(G)$. Let every component of $N(u)$ be complete and the order r of any component is at least k, where $k = 3$. Therefore, $\beta_{kst}(G) = 2$
Case (ii) Let $u \in V(G)$. Let every component of $N(u)$ be complete and the order of any component is 3, where $k > 3$. Therefore, $\beta_{kst}(G) = 2$
Conversely, Let $\beta_{kst}(G) = 2$. Let $u \in V(G)$. Then $N(u)$ can contain atmost two - k - independent elements.
Case (i) If $k = 3$. Then any component of $N(u)$ is complete and the order is atleast 3.
Case (ii) If $k > 3$. Then any component of $N(u)$ is complete and contains 3 elements. Hence the theorem.

Theorem 1.3
Let G be a path on n vertices. Then $\beta_{kst}(G) = 2$ if $|V(G)| \geq 3$ and if $|V(G)| \leq 2$,then $\beta_{kst}(G) = 1$.

Proof:
Let $G = P_n$ and let $V(G) \geq 3$. Let $V(G) = \{u_1, u_2, u_3, \cdots, u_n\}$.$\{u_1, u_3\}$ is an I_2 - set and which is dominated by u_2. Thus $\beta_{2st}(G) = 2$ and hence $\beta_{kst}(G) = 2$ (Since the degree of any vertex in P_n is either one or two). Suppose $|V(G)| \leq 2$. Then $\beta_{kst}(G) = 1$.

Theorem 1.4
Let G be a Caterpillar. $\beta_{kst}(G) = 3$ if and only if there exists a vertex on the spine of the Caterpillar which is not an end vertex and which supports a pendent vertex or there exists an end vertex which supports two pendent vertices.

Proof:
Let G be a Caterpillar. Suppose there exists a vertex on the spine of G which supports exactly one pendent vertex or there exists an end vertex of the spine which supports exactly two pendent vertices. Then $\beta_{2st}(G) = 3$ and hence $\beta_{kst}(G) = 3$.
Conversely, suppose $\beta_{kst}(G) = 3$. Then clearly either G has an internal vertex on the spine which supports exactly one pendent vertex or there exists an end vertex of the spine which support two pendent vertices. Hence the theorem.

Remark 1.2
Let G be a Caterpillar with at least three vertices. Then $\beta_{kst}(G) = 2$ if and only if no middle vertex on this spine supports a pendent vertex or the end vertices of the spine supports atmost one pendent vertex. That is, the graph is a path.

Definition 1.4
A double star is obtained by joining the centre of two stars $K_{1,r}$ and $K_{1,s}$ and it is denoted by $D_{r,s}$.

$\beta_{kst}(G)$ for some known graphs.

i. $\beta_{kst}(K_n) = \begin{cases} k - 1 \; where \; n \geq k \\ n - 1 \; if \; 2 \leq n < k \end{cases}$

ii. $\beta_{kst}(\overline{K_n}) = 1$

iii. $\beta_{kst}(K_1, n) = n$ for all $k \geq 2$

iv. $\beta_{kst}(C_n) = 2$ if $n \geq 4$ and $k \geq 2$

v. $\beta_{kst}(W_n) = \begin{cases} 3 & \text{if } 4 \leq k \leq n-1 \\ n - 1 & \text{if } k > n \end{cases}$

vi. $\beta_{kst}(P_n) = 2$ if $n \geq 3$ and $k \geq 2$

vii. $\beta_{kst}(D_{r,s}) = \max(r,s) + 1$ for all $k \geq 2$

viii. $\beta_{kst}(K_{m,n}) = \begin{cases} max(m,n) + 1 \; for \; all \; k \geq max(m,n) + 2 \\ max(m,n) \quad for \; 2 \leq k \leq max(m,n) + 1 \end{cases}$

ix. $\beta_{kst}(P) = 3$ forall$k \geq 2$

Remark 1.3
Let $G = D_{r,s}$, $r \leq s$. Then $\beta_0(G) = r + s$ and $\beta_{kst}(G) = s + 1$

Remark 1.4
Given any positive integer r, there exists a graph G such that $\beta_0(G) - \beta_{kst}(G) = r$.

Proof:
Let $G = D_{r+1,s}$ where $s \geq r + 1$. $\beta_0(G) = r + 1 + s$ and $\beta_{kst}(G) = s + 1$. Therefore, $\beta_0(G) - \beta_{kst}(G) = r$.

Corollary 1.2

$$\beta_{kst}(K(n,2)) = \begin{cases} 1 \text{ if } n = 2,3,4, & k \geq 2 \\ 3 \; if \; n = 5, & k \geq 2 \end{cases}$$

Theorem 1.5
Let G and H be two graphs. Then $\beta_{kst}(G \cup H) = \max\{\beta_{kst}(G),\ \beta_{kst}(H)\}$.

Proof:
Any kst - set of G (or H) is a kst - independent set of $(G \cup H)$. Hence the result.

Theorem 1.6
Let G and H be two graphs. Then $\beta_{kst}(G + H) = \max\{\beta_{kst}(G),\ \beta_{kst}(H)\}$

Proof:
Any kst - set of G is a I_{kst} - subset $(G + H)$and any β_{kst} - set of H is a I_{kst}, subset of $(G + H)$. Therefore, $\beta_{kst}(G + H) \geq \beta_{kst}(G),\ \beta_{kst}(H)$.
Let S be a β_{kst} - subset of $(G + H)$. Then S is a dominated by single vertex of $(G + H)$ and any k - element subset of S is disconnected in $(G + H)$. If $S \cap V(G)$ and $S \cap V(G) \neq \phi$, then any k - element subsets of S is connected, a contradiction. Therefore, $S \cap V(G) = \phi$ or $S \cap V(H) = \phi$. Therefore, $S \subseteq V(H)$ or $S \subseteq V(G)$. Therefore, $\beta_{kst}(G + H) = |S| \leq \beta_{kst}(G)$ or $\beta_{kst}(H)$. Therefore, $\beta_{kst}(G + H) = \max\{\beta_{kst}(G),\ \beta_{kst}(H)\}$

Theorem 1.7
Let G and H be two graphs. Then $\beta_{kst}(G ⍰ H) = \beta_{kst}(G) + \beta_{kst}(H)$.

Proof:

Let $S = \{v_1, v_2, v_3, \dots, v_r\}$ be a β_{kst} - set in H and let S be dominate by v in H. Let $S_1 = \{u_1, u_2, u_3, \dots, u_s\}$ be a β_{kst} - set in G with u as dominating vertex. Let $S_2 = \{(u, v_1), (u, v_2), \dots, (u, v_r), (u_1, v), (u_2, v), \dots, (u_s, v)\}$. S_2 is dominated by (u, v) in $(G \boxtimes H)$. $\{(u, v_1), (u, v_2), \dots, (u, v_r)\}$ is an I_k - set in $(G \boxtimes H)$. (Since $v_1, v_2, \dots, v_r$ is an I_k - set in H). $\{(u_1, v), (u_2, v), \dots, (u_s, v)\}$ is an I_k - set in $(G \boxtimes H)$. Clearly any k - element subsets of S_2 is disconnected. Therefore, S_2 is a I_{kst} - set of $(G \boxtimes H)$. Therefore, $\beta_{kst}(G \boxtimes H) \geq |S_2| = r + s$. Let $S_3 = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \dots, (u_r, v_r)\}$ be a β_{kst} - set of $(G \boxtimes H)$.

Let S_3 be dominated by (u', v') in $(G \boxtimes H)$. Since (u_i, v_i) is a adjacent with (u', v'), either $u_i = u', v_i$ is adjacent to v' (or) u_i is adjacent with u' and $v_i = v'$.

Therefore, $S_3 = \{(u', v_1), (u', v_2), (u', v_3), \dots, (u', v_t)(u_1, v'), (u_2, v'), (u_3, v'), \dots, (u_l, v')\}$ .Therefore, $\beta_{kst}(G \boxtimes H) = |S_3| = t + l \leq r + s$.Therefore, $\beta_{kst}(G \boxtimes H) = r + s = \beta_{kst}(G) + \beta_{kst}(H)$.

Theorem 1.8

Let H be a graph with $\beta_{kst}(H) \geq k$. Then $\beta_{kst}(C_n \boxtimes H) = \beta_{kst}(H) + 2$.

Proof:

Let $S = \{v_1, v_2, \dots, v_r\}$ be a β_{kst} - set of H, where $r \geq k$. Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$. Let S be dominated by v in $V(H)$. Since $\beta_{kst}(H) \geq k$, $v \notin S$. Let $S_1 = \{(u_1, v_1), (u_1, v_2), \dots, (u_1, v_r), (u_2, v), (u_n, v)\}$. Clearly S_1 is an I_k - set of $C_n \boxtimes H$ and S_1 is dominated by (u_1, v).

Therefore, S_1 is an I_{kst} - set of $(C_n \boxtimes H)$. Therefore, $\beta_{kst}(C_n \boxtimes H) \geq |S_1| = r + 2$. Let S_2 be a β_{kst} - set of $(C_n \boxtimes H)$. Let S_2 be dominated by (x, y) in $(C_n \boxtimes H)$. Therefore, $x \in C_n$ and $y \in H$. Therefore, $x = u_i$ for some i, $1 \leq i \leq n$.

(u_i, y) is adjacent with (u_{i-1}, y) and (u_{i+1}, y). Also if y is adjacent with t - vertices of H such that $t > r$, then $\{(u_i, w_1), (u_i, w_2), \dots, (u_i, w_t), (u_{i-1}, y), (u_{i+1}, y)\}$ is dominated by (u_i, y). If this set is an I_k - set, then $\{w_1 \dots w_t\}$ is an I_k - set of H dominated by y and $t > r$.

Therefore, $\beta_{kst}(H) > r$, a contradiction. Therefore, $t \leq r$. Therefore, $|S_2| \leq r + 2$. That is, $\beta_{kst}(C_n \boxtimes H) \leq r + 2$. Therefore, $\beta_{kst}(C_n \boxtimes H) = r + 2 = \beta_{kst}(H) + 2$.

Theorem 1.9

Let H be a graph with $\beta_{kst}(H) = k - 1$. Then $\beta_{kst}(C_n \boxtimes H) = \beta_{kst}(H) + 2$.

Proof:

Let $S = \{v_1, v_2, \dots, v_{k-1}\}$ be an β_{kst} - set of H. Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$.

Let $S_1 = \{(u, v_1), (u_1, v_2), \cdots (u_1, v_{k-1}), (u_2, v), (u_n, v)\}$, where S is dominated by v in $V(H)$. Clearly S_1 is an I_k - set of H and S_1 is dominated by (u_1, v). Therefore, $\beta_{kst}(C_n \boxtimes H) \geq k + 1$. Let S_2 be a β_{kst} - set of $C_n \boxtimes H$.

Suppose S_2 is dominated by (u_i, y), where $u_i \in V(C_n)$ and $y \in V(H)$. (u_i, y) is adjacent with (u_{i-1}, y) and (u_{i+1}, y). Also y is adjacent to t - vertices of H such that $t > k - 1$, then $\{(u_i, w_1)(u_i, w_2), \cdots, (u_i, w_t)(u_{i-1}, y)(u_i + 1, y)\}$ is dominated by (u_i, y).

If this set is an I_k - set, than $\{w_1 \cdots w_t\}$ is an I_k - set of H dominated by y and $t > k - 1$. Therefore, $\beta_{kst}(H) \geq k - 1$, a contradiction. Therefore, $t \leq k - 1$. Therefore, $|S_2| \leq k - 1 + 2 = k + 1$. Therefore, $\beta_{kst}(C_n \boxtimes H) \leq k + 1$. Therefore, $\beta_{kst}(C_n \boxtimes H) = k + 1 = \beta_{kst}(H) + 2$

Theorem 1.10

Let H be a graph with $\beta_{kst}(H) \geq k$. Then $\beta_{kst}(P_n \boxtimes H) = \beta_{kst}(H) + 2$, where $n \geq 3$.

Proof:

Let $S = \{v_1, v_2, \cdots, v_r\}$ be a β_{kst} - set of H, where $r \geq k$. Let $V(P_n) = \{u_1, u_2, \cdots, u_n\}$, $n \geq 3$. Let S be dominated by v in $V(H)$. Since $\beta_{kst}(H) \geq k$, $v \notin S$. Let $S_1 = \{(u_2, v_1), (u_2, v_2), \dots, (u_2, v_r), (u_1, v), (u_3, v)\}$, S_1 is dominated by (u_2, v). Clearly S_1 is an I_k - set of $(P_n \boxtimes H)$. Therefore, S_1 is an I_{kst} - set of $(P_n \boxtimes H)$. Therefore, $\beta_{kst}(P_n \boxtimes H) \geq r + 2 = \beta_{kst}(H) + 2$. Proceeding as in theorem (2.2.8), $\beta_{kst}(P_n \boxtimes H) \leq \beta_{kst}(H) + 2$. Therefore, $\beta_{kst}(P_n \boxtimes H) = \beta_{kst}(H) + 2$.

Theorem 1.11

Let H be a graph. Then $\beta_{kst}(P_n \boxtimes H) = \beta_{kst}(H) + 1$.

Proof:

Proceeding as in previous theorem, $S_1 = \{(u_2, v_1), (u_2, v_2), \dots, (u_2, v_r), (u_1, v)\}$ is a I_k - set of $(P_2 \boxtimes H)$ and S_1 is dominated

by (u_2, v). Therefore, $\beta_{kst}(P_2 \square H) \geq r + 1$. It can provided that $\beta_{kst}(P_2 \square H) \leq r + 1$.
Therefore, $\beta_{kst}(P_2 \square H) = r + 1 = \beta_{kst}(H) + 1$.

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