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Derivative of Adic Meromorphic Function and Their Applications

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ARTICLE INFO	ABSTRACT
Published Online: 01 April 2019 Corresponding Author: **Mohammed Mustafa** **Mobile: (+249)9118116675**	Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. Showed that by Kamal Boussaf , Alain Escassut and Jacqueline Ojeda [1] if the Wronskian of two entire functions in K is apolynomial, then both functions are polynomials. As a consequence, if meromorphic sequence of functions $f_j$on all K is transcendental and has finitely many multiple poles, then f_j' takes all values in K infinitely many times. Then study applications to meromorphic sequence of functions f_j has finitely many zeros, aproblem linkedtothe Hayman conjecture on adic field.

KEYWORDS AND PHRASES: zeros of p-adic meromorphic functions , derivative , Wronskian

Introduction and Main Results

Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $|\,.\,|$ Given $\alpha \in K$ and $(1+\epsilon) \in \mathbb{R}_+^*$ we denote by $d(\alpha,(1+\epsilon))$ the disk $\{\boldsymbol{x} \in \boldsymbol{K}_{1/2|x-\alpha|} \leq \mathbf{1}+\boldsymbol{\epsilon}\}$and by $d(\alpha,(1+\epsilon)^-)$ the disk $\left\{\boldsymbol{x} \in K_{\frac{1}{2|x-\alpha|}} \leq 1+\epsilon\right\}$ by $\mathcal{A}(K)$ the K -algebra of analytic sequence of functions in K(i.e. these to f_j power series with an infinite radius of convergence),by $\boldsymbol{\mathcal{M}(K)}$ the field of meromorphic sequence of functions in K and by $K(\boldsymbol{x})$ the field of rational functions. Given $f_j, g^j \in \mathcal{A}(K)$ denote by $\sum_j W^j(f_j, g^j)$ the Wronskian $\sum_j f_j' g^j - \sum_j f_j \acute{g}^j$.

Know that all non-constant entire sequence of functions $f_j \in \mathcal{A}(K)$ takes all values in K.
More precisely, sequence of functions $f_j \in \mathcal{A}(K)$ other than a polynomial takes all values in K infinitely many times in [2], [3] ,[4] next a non-constant meromorphic functions $f_j \in \mathcal{M}(K)$ takes every value in K, except at most one value. And more precisely, a meromorphic sequence of functions $f_j \in \mathcal{M}(K)\backslash K((x))$ takes every value in K infinitely many times except at most one value. Many previous studies were made on Picard's exceptional values for complex and a $(1+\epsilon)$-adic sequence of functions and their derivatives in [5] ,[6] and [7]. Here mean to examine precisely whether the derivative of a transcendental meromorphic sequence of function in K having finitely many multiple poles, may admit a value that is taken finitely many times and then look for applications to Hayman's problem when $m = 2$ From 4 [6], state the following Theorem A: (See e.g.e,[1])

Theorem A: Let $f_j', I_j \in \mathcal{A}(K)$ satisfy $\sum_j W^j(h_j, I_j) = c \in K$ with h_j non-affine. Then $c = 0$ and $\frac{h_j}{I_j}$ are constant.

Improve Theorem A:

Theorem 1: Let $f_j, g^j \in \mathcal{A}(K)$ be such that $\sum_j W^j(f_j, g^j)$ are non-identically zero polynomial, then both f_j, g^j are polynomials.

Remark: theorem 1 does not hold in a characteristic $\epsilon \neq -1$ indeed suppose the characteristic of K is $\epsilon \neq -1$. Let $\psi^j \in \mathcal{A}(K)$ .let $f_j = \boldsymbol{x}(\psi^j)^{(1+\epsilon)}$ and let $g^j = (\boldsymbol{x}+1)(\psi^j)^{-(1+\epsilon)}$ since $\neq 0$, we have $f_j' = (\psi^j)^{(1+\epsilon)}, \acute{g}^j = (\psi^j)^{-(1+\epsilon)}$ hence $\sum_j W^j(f_j, g^j)$ and this is true for all functions $\psi^j = \mathcal{A}(K)$

Theorem 2: Let $f_j \in \mathcal{M}(K)K(x)$ have finitely many multiple poles. Then f_j' takes every value $b \in k$ infinitely in any times.
Easily show Corollary 2.1 from Theorem 2, though it is possible to get it through an expansion in simple elements.

Corollary 2.1: Let $f_j \in \mathcal{M}(K)\backslash K(\boldsymbol{x})$. Then f_j' belongs to $\mathcal{M}(K)\backslash K(x)$ look for some applications to Hayman's problem in a $(1+\epsilon)$-adic field. Let $f_j \in M(K)$ Recall that in [8] it was shown that if m is an integer ≥ 5 or $m = 1$, then

$\sum_j f_j^{''} + f_j^{''}$ has infinitely many zeros that are not zeros of f_j. In [9] and [7] but there remain some cases where it is impossible to conclude except when the field has residue characteristic equal to zero . When m = 2, few result are known, recall also that as far as complex meromorphic functions are concerned, $\sum_j f_j^{'} + \sum_j f_j^{''}$ has infinitely many zeros that are not zeros of f_j for every but obvious counter-example, show this is wrong for m = 1 (with $f_j(\boldsymbol{x}) = e^{\boldsymbol{x}}$) and for m = 2 (with $f_j(\boldsymbol{x}) = \tan(-\boldsymbol{x})$).

Here particularly examine functions $\sum_j f_j^{'} + b\sum_j f_j^{2}$ with $b \in K^*$.

Theorem 3: let $(b^2+1) \in K^*$ and let $f_j \in \mathcal{M}(K)$ have finitely many residues at its simple poles equal to $\frac{1}{b^2+1}$ and be such that $\sum_j f_j^{'} + (b^2+1)$ has finitely many zeros, then f_j belongs to $K(\boldsymbol{x})$

Remark: $f_j(\boldsymbol{x}) = \frac{1}{x'}$ the series functions $\sum_j f_j^{'} + (b^2+1)\sum_j f_j^{2}$ has no zero whenever $b \neq 1$

Theorem 4: Let $f_j \in \boldsymbol{\mathcal{M}}(K)\backslash K(\boldsymbol{x})$ have finitely multiple zeros and let $b \in K$ then

$\sum_j \frac{f_j^{'}}{f_j^{2}} + (b^2+1)$ has infinitely many zeros. Moreover if $b \neq 0$ every zero α of

$\sum_j \frac{f_j^{'}}{f_j^{2}} + (b^2+1)$ that is not a zero of $\sum_j f_j^{'} + (b^2+1)\sum_j f_j^{2}$ are simple poles of f_j such that the residue of f at α is equal to $\frac{1}{b^2+1}$

Corollary 4.1: Let $b \in K^*$ and let $f_j \in \mathcal{M}(K)\backslash K(\boldsymbol{x})$ have finitely multiple zeros and simple poles. Then $\sum_j f_j^{'} + (b^2+1)\sum_j f_j^{2}$ has infinitely many zeros that are not zeros of f_j.

Remark: in Archimedean analysis, the typical example of a meromorphic sequence of functions f_j such that $\sum_j f_j^{'} + f_j^{2}$ has no zeros in $\tan(-x)$ and its residue is 1 at each pole of f_j. here find the same implication but n't find an example satisfying such properties 2 The proofs

Notation: Given $f_j \in \mathcal{A}(K)$ and $\epsilon > -1$, we denote by $\sum_j |f_j|(1+\epsilon)$ the norm of uniform convergence on the disk $\mathbf{1}+\boldsymbol{\epsilon}(0,1+\epsilon)$. This norm is none to be multiplicative in [10] Lemma 1: is well known in [10] :

Lemma 1: Let $f_j \in \mathcal{M}(K)$ then $\sum_j |f_j^{(k-1)}|(1+\epsilon) \leq \frac{|\sum_j f_j|(1+\epsilon)}{(1+\epsilon)^{k-1}}$ $\forall \epsilon > -1, \forall k \in N^*$

Proof of Theorem 1: First, by Theorem A: check that the claim is satisfied when $\sum_j W^j(f_j, g^j)$ is a polynomial of degree 0., suppose the claim holds when $W^j(f_j, g^j)$ are polynomial of certain degree $(1+\epsilon)$. show it for $(2+\epsilon)$..Let $f_j, g^j \in \mathcal{A}(K)$ be such that $\sum_j W^j(f_j, g^j)$ are non-identically zero polynomial $(1+\epsilon)$ of degree $(2+\epsilon)$.

By hypothesis, have $\sum_j f_j^{'} g^j - \sum_j f_j g_j^{'} = 1+\epsilon$, hence($\sum_j f_j^{''} g^j - \sum_j f_j \acute{g}^j = \left(\frac{1+\epsilon}{\epsilon}\right)$. Extract $\acute{g}^j$ and get $\sum_j \acute{g}^j = \sum_j \frac{f_j \acute{g}^j - (1+\epsilon)}{f_j}$, consider the function

$Q = \sum_j f_j^{''} \acute{g}^j - \sum_j f_j^{'} \acute{g}^j$ and replace $\acute{g}^j$ by what just found: get $Q = \sum_j f_j^{'}\left(\frac{f_j^{''} g^j - f_j \acute{g}^j}{f_j}\right) - (1+\epsilon)\sum_j \frac{f_j^{''}}{f_j}$ replace, $\sum_j f_j^{''} g^j - \sum_j f_j\, \acute{g}^j$ by $\left(\frac{1+\epsilon}{\epsilon}\right)$and obtain $Q = \sum_j \frac{f_j^{''} - (1+\epsilon) f_j^{''}}{f_j}$

thus in that expression of Q write $|Q|(1+\epsilon) \leq \sum_j \frac{|f_j|(1+\epsilon)|1+\epsilon\,|(1+\epsilon)}{(1+\epsilon)^2 |f_j|(1+\epsilon)}$ hence $|Q|(1+\epsilon) \leq \sum_j \frac{|1+\epsilon\,|(1+\epsilon)}{(1+\epsilon)^2}$ $\forall \epsilon > -1$. But by definition, Q belongs to $\mathcal{A}(K)$ and further, $\deg(\epsilon - 1)$ consequently, Q is polynomial of degree at most$(\epsilon - 1)$.

Suppose Q is not identically zero. Since $Q = \sum_j W^j(f_j^{'}, \acute{g}^j)$ and since $\deg(Q) > (1+\epsilon)$, by induction $f_j^{'}$ and $\acute{g}^j$ are polynomials and so are f_j and g^j. finally suppose $Q = 0$. Then $\left(\frac{1+\epsilon}{\epsilon}\right)\sum_j f_j^{'} - (1+\epsilon)\sum_j f_j^{''} = 0$ and therefore $f_j^{'}$ and P are two solutions of the differential equation of order 1 for meromorphic sequence of functions in K: (E)$y' = \psi^j\, y$ with $\psi^j = 1$ whereas y belongs to $\mathcal{A}(K)$. The space of solutions of (E) is known to be of dimension 0 or 1. Consequently, there exist $\lambda \in K$ such that $f_j^{'} = \lambda(1+\epsilon)$, hence f_j are polynomials, the same holds for g^j.

Proof of Theorem 2: Suppose $f_j^{'}$ has finitely many zeros. By classical results , write $f_j^{'}$ in the form $\sum_j \frac{h_j}{I_j}$ with $h_j, I_j \in \mathcal{A}(K)$, having no common zero. Consequently, all zero of $\sum_j W^j(h_j, I_j)$ are zeros f_j' except if it are multiple zeros f_j.But sinceI_j only has finitely many multiple zeros, it appears that $\sum_j W^j(h_j, I_j)$ has finitely many zeros and therefore is a polynomial. Consequently, Both h_j and I_j are polynomials a contradiction because f_j does not belong to $K(\boldsymbol{x})$, consider of $\sum_j f_j^{'} - b$ whit $b \in k$. It is derivative of $f_j - bx$ whose poles are exactly those of f_j , taking multiplicity into account, consequently $\sum_j f_j^{'} - b$ also has infinitely many zeros.

Notation: given$f_j \in K(k)$, denoted by $res_a(f_j)$ the residue of $f_j$at a.

Lemma 2: let $\sum_j f_j = \sum_j \frac{h_j}{I_j} \in \mathcal{M}(K)$ with $h_j, I_j \in \mathcal{A}(K)$ having no common zero, let $(b^2+1) \in K^*$ and $a \in K$ be a zero of $\sum_j \acute{h}_j I_j - \sum_j h_j \acute{I}_j$ that is not a zero of $\sum_j f_j^{'} + (b^2+1)\sum_j f_j^{2}$. Then a simple poles of f_j and $res_a(f_j) = \frac{1}{b^2+1}$.

Proof: Clearly, if $(a) \neq 0$, a is a zero of $\sum_j f_j^{'} + (b^2+1)\sum_j f_j^{2}$. Hence, a zero a of $\sum_j h_j^{'} I_j - \sum_j h_j \acute{I}_j + b\sum_j h_j^{2}$ that is not a zeros of $\sum_j z(\acute{f}_j + (b^2+1)\sum_j f_j^{2}$. are pole of f_j. When I_j (a) $= 0$, we have $h_j(a) \neq 0$ hence $\sum_j \acute{I}_j(a) = (b^2+1)\sum_j h_j(a) \neq 0$ and therefore a is a simple pole of f_j such that $\sum_j \frac{h_j(a)}{(I_j)'(a)} = \frac{1}{b^2+1}$ but since a is a simple pole of f_j. have $res_a \sum_j(f_j) = \sum_j \frac{h_j(a)}{(I_j)'(a)} = \frac{1}{b^2+1}$. Which ends the proof.

Proof Theorem 3 : Let $\sum_j f_j = \sum_j \frac{(1+\epsilon)}{I_j}$ with $(1+\epsilon)$ a polynomial, $I_j \in \mathcal{A}^j(K)$ having no common zero with $(1+\epsilon)$. Then $\sum_j f_j' + (b^2+1)\sum_j {f_j}^2$ $= \sum_j \frac{(\acute{h}_j)I_j - (I_j)(1+\epsilon)(1+\epsilon)(b^2+1)(1+\epsilon)^2}{{I_j}^2}$. By hypothesis, this sequence of functions has finitely many zeros, moreover if a is a zero of $\sum_j \acute{h}_j I_j - \sum_j \acute{I}_j(1+\epsilon)(b^2+1)(1+\epsilon)^2$ but is not a zeros of $\sum_j f_j'' + (b^2+1)b\sum_j {f_j}^2$ then, by lemma 2 a is a simple pole of f_j such that $res_a(f) = \frac{1}{b^2+1}$. Consequently $\left(\frac{1+\epsilon}{\epsilon}\right)\sum_j I_j - L'(1+\epsilon)(b^2+1)(1+\epsilon)^2$ has finitely many zeros and so write $\frac{\left(\frac{1+\epsilon}{\epsilon}\right)I_j - \acute{I}_j(1+\epsilon)+(b^2+1)(1+\epsilon)^2}{{I_j}^2} = \frac{Q}{{I_j}^2}$ with $Q \in K[\boldsymbol{x}]$, hence $\left(\frac{1+\epsilon}{\epsilon}\right)\sum_j I_j - \sum_j \acute{I}_j(1+\epsilon) = -(b^2+1)(1+\epsilon)^2 + Q$ but them by theorem 1: , L is a polynomial, which ends the proof.

Proof Theorem 4: Let $\sum_j g^j = \frac{f_j'}{{f_j}^2} + (b^2+1)$ .Suppose $b = 0$. Since all zeros of f_j are simple zeros except maybe finitely many, g^j has finitely many poles of order ≥ 3, hence a primitive G of g^j has finitely many multiple poles (see [11]) . Consequently, by Theorem 2 , g^j has infinitely many zeros. , Suppose $b \neq 0$, let α be zeros of g^j and let $\sum_j f_j' = \sum_j \frac{h_j}{I_j}$ with h_j , $I_j \in \mathcal{A}(K)$ having no common zero, then $\sum_j \frac{f_j'}{{f_j}^2} + (b^2+1) = \sum_j \frac{(\acute{h}_j I_j - (h_j)\acute{I}_j + (b^2+1){h_j}^2}{{h_j}^2}$ since α is a zero of $\sum_j \frac{f_j'}{{f_j}^2} + (b^2+1)$ it is not a zero of h_j and hence it is a zero of $\sum_j(\acute{h}_j I_j) - \sum_j h_j \acute{I}_j + (b^2+1)\sum_j {h_j}^2$ then lemma 2 if it is not zero of $\sum_j f_j' + (b^2+1)\sum_j {f_j}^2$ it is a simple poles of f_j such that $res_a(f_j) = \frac{1}{(b')^2+2}$ which ends the proof of theorem4.

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References

1. Kamal Boussaf, Alain Escassut, Jacqueline Ojeda : Zeros of the derivative of p-adic meromorphic functions and applications . Bull. Belg. Math. Soc. Simon , 19(2012), 367-372
2. A. Escassut : p-adic Value Distribution. Some Topicson Value Distribution and Differentiability in Complex and p-adic cAnalysis .Mathematics Monograph,. Science Press , p p 42-138 . 2008 .
3. P.C. Hu and C. C yang : Meromorphic function over non-Archimedeam fields. Kluwer Academic publishers (2000)
4. J. Ojeda :distribution de valeurs des functions meromorphes ultrametrique, applications de la theorie de Nevanlina. ,Univesite Blaise Pascal, Clermont-Ferrand, France (2008).
5. W. Bergweiler and ***X*** . C.: Pang o n the derivative of meromorphic functions unih. Multiple zeros . J. Math. Anal. Appl. 278, pp. 285-292 (2003).
6. Boussaf Picard : values of p-adic meromorphic function. p-adic Numbers, Ul- trimetric Analysis and Applications. Vol. 2, N. 4, pp. 285-292 (2010)
7. A. Escassut andJ. Ojeda : Exceptional values of p-adic analytic functions and derivatives. Complex Variables and Elliptic Equations. Vol 56, N. -4, pp. 263-269 (2011).
8. [8] J. Ojeda : Hyman’s conjecture over a p-adic field , Taiwanese Journal of mathematics. Vol. 12, N. 9, pp.2295-2313 (2008).
9. k(. Boussaf and J. Ojeda Value distribution of p-adic meromorphic functions, Bul- letin of the Belgian Mathematical Society - Simon Stevin, Vol 18, n.4, P: 667-678 (2011).
10. Escassut, A. : Analytic Elements in p-adic Analysis. World Scientific Publishing Co. Pte.Ltd. (Singapore, 1995).
11. Simon Joseph , Manal Juma and Isra Mukhtar : Applications on Triagular Subgroups of Sp with Reproducing Groups . American Association for Science and Technolpgy , International Journal of Mathematical Analysis and Applications , Vol.5 , N . 3 , pp : 66 – 84 , (2018) .