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Approximate solving method and error estimates of approximate solutions of discontinuous mixed problem for elliptic complex equations of second order in multiply connected domains

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Abstract. In this article, we discuss the approximate method of solving discontinu-ous mixed boundary value problem for nonlinear uniformly elliptic complex equation of second order in a multiply connected domain. If the complex equation and the boundary value condition satisfy certain conditions, then we can obtain some solv-ability results for the above boundary value problem by the method of parameter extension. Moreover the error estimates of approximate solutions of the discontinuous mixed problem can be obtained.

Key Words: Discontinuous mixed boundary value problem, nonlinear elliptic com-plex equations, multiply connected domains

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1. Formulation of Discontinuous Boundary Value Problems for Elliptic Complex Equations of Second Order

Let D be an $(N+1)$-connected domain with the boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j$ in $\mathbb{C}$, where $\Gamma \in C_\alpha^2$ $(0<\alpha<1)$. Without loss of generality, we assume that D is a circular domain in $|z|<1$, where the boundary consists of $N+1$ circles $\Gamma_0=\Gamma_{N+1}=\{|z|=1\}$, $\Gamma_j=\{|z-z_j|=r_j\}$, $j=1,\ldots,N$ and $z=0\in D$. In this article, the notations are as the same in References [3-14]. We discuss the nonlinear uniformly elliptic complex equation of second order

$$\begin{cases} w_{z\bar z}=F(z,w,w_z,\bar w_z,w_{zz},\bar w_{zz}),\ F=Q_1w_{zz} \\ \ldots \end{cases} \quad (1.1)$$

Suppose that the complex equation (1.1) satisfies the conditions, namely

Condition C 1) $Q_j(z,w,w_z,\bar w_z,X,Y)(j=1,2)$, $A_j(z,w,w_z,\bar w_z)(j=1,\cdots,4)$ are measurable in $z\in D$ for all continuously differentiable functions $w(z)$ in D and all mea-surable functions $X(z),\ Y(z)\in L_{p_0}(\bar D)$, and satisfy

$$L_p[A_j(z,w,w_z,\bar w_z),\bar D]\le k_{j-1},\ j=1,\cdots,4, \quad (1.2)$$

where $p_0,\ p\ (2<p_0\le p),\ k_j(j=0,1,2,3)$ are non-negative constants.

2) The above functions are continuous in $w, w_z, \bar{w}_z \in \mathbb{C}$ for almost every point $z \in D$, $X, Y \in \mathbb{C}$, and $Q_j = 0 (j = 1, 2)$, $A_j = 0 (j = 1, \cdots, 4)$ for $z \notin D$. Besides, we assume that $Q_2 = 0$ in a neighborhood of Γ.

3) The complex equation (1.1) satisfies the following uniform ellipticity condition, namely for any functions $w(z) \in C^1(D)$ and $X_j, Y_j \in \mathbb{C} (j = 1, 2)$, the inequality

$$|F(z, w, w_z, \bar{w}_z, X_1, Y_1) - F(z, w, w_z, \bar{w}_z, X_2, Y_2)| \le q_1|X_1 - X_2| + q_2|Y_1 - Y_2|,$$

holds for almost every point $z \in D$, where $q_j (q_1 + q_2 < 1)(j = 1, 2)$ are all non-negative constants.

We introduce the discontinuous mixed boundary value problem for the second order complex equation (1.1) namely

Problem M Find a continuously differentiable solution $w(z)$ in $D^* = \bar{D} \setminus Z$ of complex equation (1.1) satisfying the boundary conditions

$$\mathrm{Re}[\overline{\lambda_1(z) w_z} + \sigma(z) w(z)] = \tau_1(z), \ \mathrm{Re}[\overline{\lambda_2(z) w(z)}] = \tau_2(z), \ z \in \Gamma^* = \Gamma \setminus Z, \tag{1.4}$$

where $Z = \{t_1, t_2, \ldots, t_m\}$ are the first kind of discontinuous points of $\lambda_j(z) (j = 1, 2)$ on Γ, Γ_l is an arc from the point t_{l-1} to t_l on Γ, $\Gamma_l (l = 1, 2, \ldots, m)$ does not include the end points. We can assume that $t_l \in \Gamma_0 (l = 1, \ldots, m_0)$; $t_l \in \Gamma_1 (l = m_0 + 1, \ldots, m_1)$; $\ldots$; $t_l \in \Gamma_N (l = m_{N-1} + 1, \ldots, m)$ are all discontinuous points of $\lambda_j(z)$ on Γ. Denote by $\lambda_j(t_l - 0)$ and $\lambda_j(t_l + 0)$ the left limit and right limit of $\lambda_j(z)$ as $z \to t_l (\in \Gamma, l = 1, 2, \ldots, m; j = 1, 2)$, and

$$e^{i\phi_{jl}} = \frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)}, \ \gamma_{jl} = \frac{1}{\pi i} \ln \frac{\lambda_j(t_l - 0)}{\lambda_j(t_l + 0)} = \frac{\phi_{jl}}{\pi} - K_{jl},$$

$$K_{jl} = \left[\frac{\phi_{jl}}{\pi}\right] + J_{jl}, \ J_{jl} = 0 \text{ or } 1, \ l = 1, \ldots, m; \ j = 1, 2, \tag{1.5}$$

in which $0 \le \gamma_{jl} < 1$ when $J_{jl} = 0$, and $-1 < \gamma_{jl} < 0$ when $J_{jl} = 1$, $j = 1, 2$, $l = 1, \ldots, m$. There is no harm in assuming that the partial indexes K_{jk} are denoted the partial index on $\Gamma_k (k = 1, \ldots, N_0 (\le N))$ are not integers, and the partial indexes $\tilde{K}_{jk} = \sum_{l = m_{k-1}+1}^{m_k} K_{jl}$ of $\lambda_j(z)$ on $\Gamma_k (k = 0, N_0 + 1, \ldots, N)$ are integers. Set

$$K_j = \frac{1}{2\pi} \Delta_\Gamma \arg \lambda_j(z) = \sum_{l=1}^{m} \frac{K_{jl}}{2}, \ j = 1, 2,$$

and $K = (K_1, K_2)$ is called the index of Problem M. Moreover, $\lambda_j(z), r_j(z) (j = 1, 2), \sigma(z)$ satisfies the conditions

$$C_\alpha^{j-1}[\lambda_j, \hat{\Gamma}_l] \le k_0, \ C_\alpha[|z - t_{l-1}|^{\beta_{jl-1}} |z - t_l|^{\beta_{jl}} \sigma(z), \hat{\Gamma}_l] \le k_4,$$

$$C_\alpha^{j-1}[|z - t_l|^{\beta_{jl}} |z - t_{l-1}|^{\beta_{jl-1}} \tau_j(z), \hat{\Gamma}_l] \le k_5, \ j = 1, 2, \ l = 1, \ldots, m, \tag{1.6}$$

in which $\alpha (1/2 < \alpha < 1)$, $k_j (j = 0, 4, 5)$ are non-negative constants, $\beta_{jl} + \gamma_{jl} < 1$, $l = 1, \ldots, m$, $j = 1, 2$; we require that the solution $[w_1(z), w_2(z)]$ possesses the property

$$R(z) w_z, \ R(z) \bar{w}_z \in C_\delta(\bar{D}), \ R(z) = \prod_{l=1}^{m} |z - t_l|^{\eta_l/\tau^2}, \ \eta_l = \max(\eta_{1l}, \eta_{2l}),$$

$$\eta_{jl} = \begin{cases} \beta_{jl} + \tau, & \text{for } \gamma_{jl} \ge 0, \text{ and } \gamma_{jl} < 0, \ \beta_{jl} \ge |\gamma_{jl}|, \\ |\gamma_{jl}| + \tau, & \text{for } \gamma_{jl} < 0, \ \beta_{jl} < |\gamma_{jl}|, \ l = 1, \ldots, m, \end{cases} \quad j = 1, 2 \tag{1.7}$$

in the neighborhood $(\subset D)$ of $t_l (l = 1, \ldots, m)$, where $\delta, \tau (< \min(\alpha, 1 - 2/p_0))$ are small positive constants. In general, Problem M may not be solvable. Hence we consider the modified well-posed-ness of Problem M as follows.

Problem N Find a continuously differentiable solution $w(z)$ in D^* of the complex system (1.1) satisfying the modified boundary conditions

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)w_z} + \sigma(z)w(z)] = \tau_1(z) + h_1(z), \\ \mathrm{Re}[\overline{\lambda_2(z)w(z)}] = \tau_2(z) + h_2(z), \end{cases} \quad z \in \Gamma^*, \qquad (1.8)$$

where

$$\frac{h_j(z)\lambda_j(z)}{X_j(z)} = \begin{cases} 0, \ z \in \Gamma_0, & \\ h_{jk}, \ z \in \Gamma_k, \ k = 1, \ldots, N, & \text{if } K_j \geq 0, \\ h_{jk}, \ z \in \Gamma_k, \ k = 1, \ldots, N, & \\ [1 + (-1)^{2K_j}] h_{j0} + \mathrm{Re} \sum_{m=1}^{[|K_j|+1/2]-1} (h^+_{jm} + i h^-_{jm}) z^m, \ z \in \Gamma_0, & \text{if } K_j < 0, \\ j = 1, 2, & \end{cases}$$

in which $X_j(z)(j = 1, 2)$ are the solutions of some Dirichlet problems in D, $h_{jk}(k = [1 - (-1)^{2K_j}]/2, \ldots, N)$, h^+_{jm}, h^-_{jm} $(m = 1, \ldots, [|K_j|+1/2] - 1, j = 1, 2)$ are unknown real constants to be determined appropriately. In addition, for $K_j \geq 0$ $(j = 1, 2)$ the solution $w(z)$ is assumed to satisfy point conditions

$$\begin{aligned} &\mathrm{Im}[\overline{\lambda_1(a_l)U(a_l)} + \sigma_1(a_l)w(a_l)] = b_{jl}, \ l \in J_1, \\ &\mathrm{Im}[\overline{\lambda_2(a_l)V(a_l)}] = b_{2l}, \ l \in J_2, \\ &J_j = \{1, \ldots, 2K_j + 1\}, \ K_j \geq 0, \ j = 1, 2, \end{aligned} \qquad (1.9)$$

where $a_l \in \Gamma_0$ $(l \in J_j)$ are distinct points; and $b_{jl}(l \in J_j, j = 1, 2)$ are all real constants satisfying the conditions

$$|b_{jl}| \leq k_6, \ l \in J_j, \ j = 1, 2, \qquad (1.10)$$

with the positive constant k_6.

The well posed-ness is a generalization of corresponding problem of the Riemann-Hilbert problem for first order elliptic complex equations (see [3]), which is not a simple problem, hence it is not easy to understand. Moreover, Problem N with $A_4 = 0, G = 0, \tau_j(z) = 0, b_{jl} = 0$ $(l \in J_j, j = 1, 2)$ is called Problem N_0.

2. Estimates of Solutions of Discontinuous Boundary Value Problems for Elliptic Complex Equations of Second Order

First of all, we give the corresponding complex system of complex equations in the form

$$\begin{aligned} &U_{\bar z} = F(z, w, U, V, U_z, V_z), \ F = Q_1 U_z + Q_2 V_z \\ &+ A_1 U + A_2 V + A_3 w + A_4, \ V_{\bar z} = \overline{U_z} = \bar w_{zz1}, \end{aligned} \qquad (2.1)$$

where $U = w_z, V = \bar w_z$.

Theorem 2.1 *Let the complex equation* (1.1) *satisfy Condition* C. *Then any solution* $w(z)(RSw_{\bar z\bar z} \in L_{p_0}(\bar D), 2 < p_0 \leq p)$ *of Problem* N *for* (2.1) *possesses the representation*

$$w(z) = \Phi_2(z) + T[\Phi_1 + T\rho] = \Psi(z) + TT\rho, \ T\rho = -\frac{1}{\pi}\iint_D \frac{\rho(\zeta)}{\zeta - z} d\sigma_\zeta, \qquad (2.2)$$

where ½(z) = $w_{z^1z^1}$; $©_j(z)$ (j = 1; 2) are analytic functions in D; ª(z) = $\overline{©_2(z)}$ + $\overline{T}$ $©_1$ is a complex function in D, and w_z = $©_1(z)$ + T½; w(z) = ª(z) + $T\overline{T½}$ satisfy the boundary conditions

$$\begin{cases} \mathrm{Re}[\overline{¸_1 z)}(©_1(z)+T½)] = ¡\mathrm{Re}[¾(z)w]+¿_1(z)+h_1(z);\ z\ 2\ ¡¤; \\ \mathrm{Re}[¸_2(z)(ª(z) + T\overline{T½})] = ¿_2(z) + h_2(z); \end{cases} \quad (2{:}3)$$

and point conditions

$$\begin{cases} \mathrm{Im}[\overline{¸_1(z)}(©_1(z)+T½)]j_{z=a_j} = ¡\mathrm{Im}[¾(a_j)w(a_j)]+b_{1j};\ j\ 2J_1; \\ \mathrm{Im}[¸_2(z)(ª(z) + T\overline{T½})]j_{z=a_j} = b_{2j};\ j\ 2\ J_2: \end{cases} \quad (2{:}4)$$

Proof Let the solution w(z) of Problem N be substituted into the equation (2.1) and denote the equation in the form

$$w_{zz^1} = ½(z);\ R(z)S(z)½(z)\ 2\ L_{p0}\ (\overline{D}); \quad (2{:}5)$$

hence we have

$$w_z = U(z) = ©_1(z) + T½: \quad (2{:}6)$$

Noting that w(z) satisfies the second formulas of boundary and point conditions (1.8) and (1.9), it is easy to see that w_z = $©_1(z)$ + T½ satisfies the complex equation

$$w_{zz} = ©_1{}^0(z) + ¦½ \text{ in } D; \quad (2{:}7)$$

and the boundary condition

¡

where s is the arc length parameter of ¡.

Theorem 2.2 *Suppose that Condition* C *holds and* q_2; k_1; k_2; k_4 *in Condition* C *and* (1:2); (1:3); (1:6) *are small enough. Then any solution* w(z)(RSw_{zz^1} = RS½(z) 2 L_{p0} (D)) *of Problem* N *for* (2:1) *with* G(z; w; U; V) = 0 *satisfies the estimates*

$$\hat{S}_1 w = C_{\bar{¯}}[w(z);\ \overline{D}] = C_{\bar{¯}}[R^{1\ 0}(z)w(z);\ \overline{D}] \cdot M_1 k^{¤}_1; \quad (2{:}9)$$

$$\hat{S}_2 w = L_{p0}[w(z);\ \overline{D}] = L_{p0}[RS(jw_{zz^1}j+jw_{zz}j+j\bar{w}_{zz}j);\ \overline{D}] \cdot M_2 k^{¤}; \quad (2{:}10)$$

in which $S(z) = ¦^m_{l=1} jz ¡ t_l j^{1=¿2}$; ¯ = min(®; 1¡2=$p_0$); M_j = M_j(q_1; p_0; k_0; ®; K; D) (j = 1; 2) *are non-negative constants, and* $k^¤$ = k_3 + k_5 + k_6:

Proof Let the solution w(z) of Problem N be substituted into the equation (2.1) and boundary conditions (1.8),(1.9). It is easy to see that w(z) satisfies the complex equation (2.1) and the second formulas in (1.8) and (1.9), i.e.

$$w_{zz^1} ¡Q_1 w_{zz} ¡A_1 w_z = A+A_4;\ A=Q_2\bar{w}_{zz} + A_2\bar{w}_z + A_3 w \text{ in } D; \quad (2{:}11)$$

$$\begin{cases} \mathrm{Re}[\overline{\ \ } w_z] = r(z) + ¿_1(z) + h_1(z);\ r = ¡\mathrm{Re}[¾(z)w] \text{ on } ¡; \\ \mathrm{Im}[\ldots] \end{cases} \quad (2{:}12)$$

According to the method in the proof of Theorem 1.2.3, Chapter I, [13], we can derive that the solution w_z of the boundary value problem (2.11)–(2.13) satisfies the estimates

$$C_{\bar{¯}}[R(z)w_z;\ \overline{D}] \cdot M_3 k_¤; \quad (2{:}14)$$

$$L_{p0}[RS(jw_{zz}j + j\bar{w}_{zz}j);\ D] \cdot M_4 k_¤; \quad (2{:}15)$$

where $^{-}; p_0$ are stated as before, and $M_j = M_j(q_1; p_0; k_0; ®; K; D)(j = 3; 4)$ are non-negative constants,

$$k_{¤} = q_2 L_{p_0}[RSw_{zz}; \overline{D}] + k_1 C[Rw_z; \overline{D}] + k_2 C[R^0(z)w(z); \overline{D}] + C_{®}[r; ¡] + \textstyle\sum_{j \, 2 J_1}[js_j j + jb_{1j} j]$$

$$\cdot \ q_2 S_2 w + (k_1 + k_2 + k_4) S_1 w + k^{¤} \ \cdot \ q_2 S_2 w + k^{¤¤} S_1 w + k^{¤};$$

in which $k^{¤¤} = k_1 + k_2 + k_4$. Moreover $w(z)$ satisfies (2.6) and the second formulas in (1.8) and (1.9), and we can obtain the estimates

$$C_{-}[R^0(z)w(z); \overline{D}] \cdot M_5 k_{¤¤}; \qquad (2{:}16)$$

$$L_{p_0}[R(jw_z j + j\overline{w}_z j); \overline{D}] \cdot M_6 k_{¤¤}; \qquad (2{:}17)$$

where

$$k_{¤¤} = C_{-}[Rw_z; \overline{D}] + k_5 + k_6 \cdot M_3 k_{¤} + k^{¤}$$

$$\cdot \ M_3[q_2 S_2 w + k^{¤¤} S_1 w + k^{¤}] + k^{¤}$$

$$\cdot \ M_3[q_2 S_2 w + k^{¤¤} S_1 w] + k^{¤}(1 + M_3):$$

In addition, from (2.5),(2.6) and the second formulas in (1.8) and (1.9), we know that w_z is a solution of the equation

$$w_{zz^1} = ©_{1z}(z) + ¦½; \ z \, 2 \, D; \qquad (2{:}18)$$

satisfying the boundary condition

$$z^{\upsilon} \qquad i \qquad 2$$

Here we mention that Condition C and (1.6),(1.10) can be derived the function $©_{1z}(z) \, 2 \, L_{p_0}(\overline{D})$ $(p_0 > 2)$ by (3.6), Chapter I, [1]. Thus we can get that $w_z = {}^{a}{}_{1z}(z) + ¦T½$ satisfies

$$C_{-}[Rw_z; \overline{D}] \cdot M_7[L_{p_0}[RSw_{zz}; \overline{D}] + k_0(C_{-}[Rw_{z^1}; ¡] + C_{-}[R^0(z)w(z); ¡]) + k^{¤}] \qquad (2{:}20)$$

$$\cdot M_7 f[M_4 + k_0 M_3(1 + M_5)][q_2 S_2 w + k^{¤¤} S_1 w + k^{¤}] + (1 + k_0 M_3 M_5) k^{¤} g;$$

where $M_7 = M_7(q_1; p_0; k_0; ®; K; D)$. Thus the estimates

$$S_1 w = C_{-}[w(z); \overline{D}] + C_{-}[Rw_z; \overline{D}] + C_{-}[R\overline{w}_z; \overline{D}]$$

$$\cdot M_7 f[M_4 + k_0 M_3(1 + M_5)][q_2 S_2 w + k^{¤¤} S_1 w + k^{¤}]$$

$$+ (1 + k_0 M_3 M_5) k^{¤} g + M_3 k_{¤} + M_5 k_{¤¤} \qquad (2{:}21)$$

$$\cdot M_7[M_4 + k_0 M_3(1 + M_5) + M_3(1 + M_5)][q_2 S_2 w + k^{¤¤} S_1 w + k^{¤}]$$

$$+ [M_7 + M_3 M_5(1 + k_0 M_7)] k^{¤};$$

and

$$L_{p_0}[RSw_{zz}; \overline{D}] \cdot M_8 f L_{p_0}[RS\overline{w}_{zz}; \overline{D}] + k_0(C_{-}[Rw_z; ¡] + C_{-}[R^0(z)w(z); ¡]) + k_5 + k_6 g$$

$$\cdot \ M_8 f[M_4 + k_0 M_3(1 + M_5)][q_2 S_2 w + k^{¤¤} S_1 w + k^{¤}] + (1 + k_0 M_5) g k^{¤}; \qquad (2{:}22)$$

can be derived, where $M_8 = M_8(q_1; p_0; k_0; ®; K; D)$. Combining (2.14) –(2.17) and (2.21), (2.22), we obtain

$$S_1w + S_2w \cdot (M_7 + M_8)f[M_4 + k_0M_3(1 + M_5) + M_3(1 + M_5)]$$

$$£[q_2S_2w+k^{¤¤}S_1w+k^{¤}]+k^{¤}g+M_5(M_3 +k_0M_3M_7 +k_0M_8))]k^{¤}+M_4k_{¤}$$

$$\cdot fM_4 + (M_7 +M_8)[M_4 + k_0M_3(1 + M_5) + M_3(1+M_5)]g$$

$$£[q_2S_2w+k^{¤¤}S_1w+k^{¤}]+[M_7 +M_8 +M_5(M_3 +k_0M_3M_7 +k_0M_8]k^{¤} \qquad (2:23)$$

$$\cdot \; fM_4 + (M_7 + M_8)[M_4 + M_3(1 + k_0)(1 + M_5)]g(q_2 + k^{¤¤})$$

$$£[S_1w+S_2w]+fM_4 + (M_7 + M_8)[1+M_4 + M_3(1 + k_0)(1 + M_5)]$$

$$+M_5(M_3 +k_0M_3M_7 +k_0M_8)gk^{¤} \cdot M_9k^{¤} =M_9(k_3 +k_5 +k_6):$$

Because we choose the sufficiently small positive constants $q_2; k_1; k_2; k_4$ in Condition C and (1.2),(1.3),(1.6), such that

$$1¡ fM_4 +(M_7 +M_8)[M_4 +M_3(1 + k_0)(1 + M_5)](q_2 +k^{¤¤})g > 1=2;$$

and can select the positive constant $M_9 = 2fM_4+(M_7+M_8)[1+M_4+M_3(1 + k_0)(1 + M_5)]+$ $M_5(M_3+k_0M_3M_7+k_0M_8)g:$ Thus the estimates (2.9) and (2.10) with $M_1 = M_2 = M_9$ are derived.

— In order to prove the uniqueness of solutions of Problem N for (1.1), we need to add the following condition: For any continuously differentiable functions $w_j(z)(j = 1; 2)$ in D and any continuous functions $U(z); V(z) \; 2 \; W_p^1{}_0(D)(2 < p_0 \cdot p)$, there is

$$F(z; w_1; w_{1z}; \bar{w}_{1z}; U_z; V_z)¡F(z; w_2; w_{2z}; \bar{w}_{2z}U_z; V_z)$$
$$= \tilde{Q}_1U_z + \tilde{Q}_2V_z + \tilde{A}_1(w_{1z} ¡\underline{w}_{2z}) + \tilde{A}_2(\bar{w}_{1z} ¡\bar{w}_{2z}) + \tilde{A}_3(w_1 ¡w_2); \qquad (2:24)$$

where $j\tilde{Q}_j \cdot q_j; j = 1; 2; \tilde{A}_j 2 L_{p0}(D); j = 1; 2; 3$. In particular, if the equation (1.1) is a linear complex equation, then (2.24) is obviously held, namely

$$F(z; w_1; w_{1z}; \bar{w}_{1z}; U_z; V_z) ¡ F(z; w_2; w_{2z}; \bar{w}_{2z}U_z; V_z)$$
$$= A_1(z)(w_{1z} ¡w_{2z})+A_2(z)(\bar{w}_{1z} ¡\bar{w}_{2z})+A_3(z)(w_1 ¡w_2); \qquad (2:25)$$

where $L_p[A_j(z); D] \cdot k_{j¡1}; j = 1; 2; 3$.

Theorem 2.3 *If Condition* C *and* (2:27) *hold, and* $q_2; k_1; k_2; k_4$ *in* (1:2); (1:3); (1:6) *are small enough, then the solution* $[w(z); U(z); V(z)]$ *of Problem* N *for* (2:1) *is unique.*

Proof Denote by $[w_j(z); U_j(z); V_j(z)](j = 1; 2)$ two solutions of Problem N for (1.1) and substitute them into (2.1),(1.8) and (1.10), we see that $[w; U; V] = [w_1(z) ¡w_2(z); U_1(z) ¡ U_2(z); V_1(z) ¡ V_2(z)]$ is a solution of the following homogeneous boundary value problem

$$U_{\bar{z}} = \tilde{Q}_1U_z + \tilde{Q}_2V_z + \tilde{A}_1U + \tilde{A}_2\bar{V} + \tilde{A}_3w; \; V_{\bar{z}} = U_z; \qquad (2:26)$$

$$\begin{cases} \mathrm{Re}[\overline{¸_1(z)}U(z) + ¾(z)w(z)] = h_1(z); \\ \mathrm{Re}[\overline{¸_2(z)w(z)}] = h_2(z); & z \; 2 \; ¡; \end{cases} \qquad (2:27)$$

$$\begin{cases} \mathrm{Im}[\overline{¸_1(z)}U(z) + ¾(z)w(z)]_{z=a_j} = 0; \; j \; 2 \; J_1; \\ \mathrm{Im}[\overline{¸_2(z)w(z)}]_{z=a_j} = 0; \; j \; 2 \; J_2; \end{cases} \qquad (2:28)$$

$$w(z) = \int_{a_1}^{z} [U(z)dz + V(z)d\bar{z}], \qquad (2.29)$$

the coefficients of which satisfy same conditions, provided q_2, k_1, k_2 and k_4 are sufficiently small, from Theorem 2.2, we can derive that $w(z) = U(z) = V(z) = 0$ in D, i.e. $w_1(z) = w_2(z), U_1(z) = U_2(z), V_1(z) = V_2(z)$ in D.

3. Estimates of Solutions for Modified Problem of Elliptic System of First Order Equations

In this section, we mainly discuss the modified mixed boundary value problem N for nonlinear elliptic system of second order equations in the complex form as stated in (1.40) of Chapter 1, [7], i.e.

$$\begin{cases} w_{z\bar{z}} = F(z, w, w_z, w_{\bar{z}}, w_{zz}, w_{\bar{z}\bar{z}}), \ F = Q_1 w_{zz} + Q_2 w_{\bar{z}\bar{z}} \\ +A_1 w_z + A_2 w_{\bar{z}} + A_3 w + A_4, \ Q_j = Q_j(z, w, w_z, w_{\bar{z}}, \\ w_{zz}, w_{\bar{z}\bar{z}}), \ j = 1, 2, \ A_j = A_j(z, w, w_z, w_{\bar{z}}), \ j = 1, \cdots, 4, \end{cases} \qquad (3.1)$$

with the modified boundary conditions

$$\begin{cases} \mathrm{Re}[\overline{\lambda_1(z)} w_z + \sigma(z) w(z)] = \tau_1(z) + h_1(z), \\ \mathrm{Re}[\overline{\lambda_2(z) w(z)}] = \tau_2(z) + h_2(z), \ z \in \Gamma, \end{cases} \qquad (3.2)$$

$$\begin{cases} \mathrm{Im}[\overline{\lambda_1(z)} w_z + \sigma(z) w(z)]|_{z=a_j} = b_{1j}, \ j \in J_1, \\ \mathrm{Im}[\overline{\lambda_2(z) w(z)}]|_{z=a_j} = b_{2j}, \ j \in J_2, \end{cases} \qquad (3.3)$$

where $\lambda_j(z), \sigma(z), \tau_j(z), h_j(z), a_j, b_{jl}(l \in J_j, j = 1, 2)$ are as stated in (1.8)–(1.11) of Sec-tion 1. Suppose that (3.1) satisfies Condition C and the following condition:

$$F(z, w_1, w_{1z}, w_{1\bar{z}}, U, V) - F(z, w_2, w_{2z}, w_{2\bar{z}}, U, V) = \tilde{Q}_1 U_z + \tilde{Q}_2 V_{\bar{z}} + \tilde{A}_1 (w_1 - w_2)_z$$
$$+ \tilde{A}_2 (\bar{w}_1 - \bar{w}_2)_{\bar{z}} + \tilde{A}_3 (w_1 - w_2), \ |\tilde{Q}_j| \le q_j, \ j = 1, 2, \ L_{p_0}(\tilde{A}_j, \bar{D}) \le k_{j-1} \le k_0, \ j = 1, 2, 3 \qquad (3.4)$$

for any continuously differentiable functions $w_1(z), w_2(z)$ and any measurable functions $U(z), V(z)$ on D, where $p_0 (2 < p_0 \le p), k_j (j = 0, 1, 2)$ are nonnegative constants.

Firstly, we prove the existence of solutions of Problem N for (3.1) by using the method of parameter extension.

Theorem 3.1 *Let the nonlinear complex equation* (3.1) *satisfy Condition* C, (3.4) *and the constants* $q_2, k_1, k_2, k_4, k_5, k_6$ *in Section* 1 *and* (3.4) *are small enough. Then Problem* N *for* (3.1) *is solvable.*

Proof Let us introduce a complex equation with the parameter $t \in [0, 1]$:

$$w_{z\bar{z}} = tF(z, w, w_z, w_{\bar{z}}, w_{zz}, w_{\bar{z}\bar{z}}) + A(z), \ R(z)S(z)A(z) \in L_{p_0}(\bar{D}). \qquad (3.5)$$

When $t = 0$, it can be found a unique solution $w(z)$ of Problem N for the simple complex equation $w_{z\bar{z}} = A(z)$ by the Newton imbedding method. In fact, we may consider the following boundary value problem N with the parameter $t\ 2\ [0;\ 1]$:

$$
8\quad w_{z\bar{z}^1} = tA(z);\ 0 \cdot t \cdot 1;
$$

$$
\mathrm{Re}[\overline{\lambda_2(z)w(z)}] = \varrho_2(z) + h_2(z);\ z\ 2\ \Gamma; \qquad (3.6)
$$

$$
\mathrm{Im}[\overline{\lambda_1(z)}w_z + \sigma(z)w(z)]|_{z=a_j} = b_{1j};\ j\ 2\ J_1;
$$

$$
\mathrm{Im}[\overline{\lambda_2(z)}w(z)]_{z=a_j} = b_{2j};\ j\ 2\ J_2:
$$

Obviously, (3.6) with $t = 0$ possesses a solution of Problem N. From this, we can derive the solvability of Problem N for (3.6) with $t = 1$. Suppose that Problem N for the complex equation (3.5) with $t = t_0(0 \cdot t_0 \cdot 1)$ is solvable. To prove that there exists a positive constant $\pm$, so that Problem N of (3.5) for every $t\ 2\ E = fjt\ ¡\ t_0j \cdot \pm;\ 0 \cdot t \cdot 1g$ and any $RSA(z)\ 2\ L_{p0}(D)$ is solvable. We rewrite (3.5) in the form

$$
w_{z\bar{z}^1}\ ¡\ t_0F(z;\ w;\ w_z;\ \bar{w}_z;\ w_{zz};\ \bar{w}_{zz}) = (t\ ¡\ t_0)F(z;\ w;\ w_z;\ \bar{w}_z;\ w_{zz};\ \bar{w}_{zz}) + A(z): \qquad (3.7)
$$

Choosing an arbitrary function $w_0(z)\ 2\ B = \hat{C}^1(\bar{D}) \cap \hat{L}^2_{p0}(D);$ there is no harm in assuming $w_0(z) = 0$, and substituting $w_0(z)$ into the position of $w(z)$ in the right hand side of (3.7), we denote by $w_1(z)$ the solution of (3.7). Using the successive iteration, we find a sequence of solutions: $w_n(z)\ 2\ B;\ n = 1;\ 2;\ ¢¢¢\ ;$ which satisfy

$$
\begin{aligned}
&w_{n+1z\bar{z}^1}\ ¡\ t_0F(z;\ w_{n+1};\ w_{n+1z};\ \bar{w}_{n+1z};\ w_{n+1zz};\ \bar{w}_{n+1zz}) \\
&= (t\ ¡\ t_0)F(z;\ w_n;\ w_{nz};\ \bar{w}_{nz};\ w_{nzz};\ \bar{w}_{nzz}) + A(z):
\end{aligned} \qquad (3.8)
$$

From (3.8) it follows

$$
\begin{aligned}
&(w_{n+1}\ ¡\ w_n)_{z\bar{z}^1}\ ¡\ t_0g(w_{n+1};\ w_n) = (t\ ¡\ t_0)g(w_n;\ w_{n¡1}); \\
&g(w_{n+1};\ w_n) = F(z;\ w_{n+1};\ w_{n+1z};\ \bar{w}_{n+1z};\ w_{n+1zz};\ \bar{w}_{n+1zz}) \\
&\qquad ¡\ F(z;\ w_n;\ w_{nz};\ \bar{w}_{nz};\ w_{nzz};\ \bar{w}_{nzz}):
\end{aligned} \qquad (3.9)
$$

By Condition C, it is easy to see that

$$
\begin{aligned}
&g(w_{n+1};\ w_n) = \tilde{Q}_1(w_{n+1}\ ¡\ w_n)_{zz} + \tilde{Q}_2(\bar{w}_{n+1}\ ¡\ \bar{w}_n)_{zz} \\
&+\tilde{A}_1(w_{n+1}\ ¡\ w_n)_z + \tilde{A}_2(\bar{w}_{n+1}\ ¡\ \bar{w}_n)_z + \tilde{A}_3(w_{n+1}\ ¡\ w_n); \\
&j\tilde{Q}_jj \cdot q_j;\ j = 1;\ 2;\ L_{p0}[\tilde{A}_j;\ \bar{D}] \cdot k_{j¡1} \cdot k_0;\ j = 1;\ 2;\ 3;
\end{aligned}
$$

where $q_j(j = 1;\ 2);\ k_j(j = 0;\ 1;\ 2)$ are nonnegative constants satisfying the condition $q_1 + q_2 < 1$. Hence

$$
\begin{aligned}
L_{p0}[RSg(w_{n+1};\ w_n);\ \bar{D}] &\cdot (q_1 + q_2)L_{p0}[RS(j(w_{n+1}\ ¡\ w_n)_{zz}j + j(\bar{w}_{n+1}\ ¡\ \bar{w}_n)_{zz}j);\\
\bar{D}] + k_0C^1[R^0(w_{n+1}\ ¡\ w_n);\ \bar{D}] &\cdot (q_1 + q_2 + k_0)S(w_n\ ¡\ w_{n¡1});
\end{aligned}
$$

where

$$S(w_{n+1} - w_n) = C^{-1}[R(w_{n+1} - w_n), \overline{D}] + L_{p_0}[RS(|(w_{n+1} - w_n)_{z\bar{z}}|$$
$$+|(w_{n+1} - w_n)_{zz}| + |(\overline{w_{n+1} - w_n})_{zz}|), D],$$
$$C^{-1}[R^0(w_{n+1} - w_n), \overline{D}] = C^{-}[R^0(w_{n+1} - w_n), \overline{D}]+$$
$$+C^{-}[R(w_{n+1} - w_n)_z, D] + C^{-}[R(\overline{w_{n+1} - w_n})_z, D].$$

Moreover, $w_{n+1} - w_n$ satisfies the homogenous boundary conditions

$$\operatorname{Re}[\overline{\lambda_1(z)}(w_{n+1} - w_n)_z + \sigma(z)(w_{n+1} - w_n)] = h_1(z),$$
$$\operatorname{Re}[\overline{\lambda_2(z)}(w_{n+1} - w_n)] = h_2(z),\ z \in \Gamma, \tag{3.10}$$

$$\operatorname{Im}[\overline{\lambda_1(z)}(w_{n+1} - w_n)_z + \sigma(z)(w_{n+1} - w_n)]|_{z=a_j} = 0,\ j \in J_1,$$
$$\operatorname{Im}[\lambda_2(z)(w_{n+1} - w_n)]|_{z=a_j} = 0,\ j \in J_2. \tag{3.11}$$

On the basis of Theorem 5.6, Chapter 1, [7], we have the estimate

$$S(w_{n+1} - w_n) \le M|t - t_0|(q_1 + q_2 + k_0)S(w_n - w_{n-1}), \tag{3.12}$$

where $M = M_{14}(q_0, p_0, k_0, \alpha, K, D)(K = (K_1, K_2))$ is a constant as stated in Theorem 5.6 of Chapter 1. Choosing that a positive number δ is sufficiently small so that $\eta =$ $\delta M(q_1 + q_2 + k_0) < 1$, it can be obtained that when $t \in E$,

$$S(w_{n+1} - w_n) \le \eta S(w_n - w_{n-1}) = \eta^n S(w_1).$$

Thus

$$S(w_n - w_m) \le (\eta^{n-1} + \eta^{n-2} + \cdots + \eta^m)S(w_1) \le \frac{\eta^{N+1}}{1 - \eta} S(w_1)$$

for $n \ge m > N$, where N is a positive integer. This shows $S(w_n - w_m) \to 0$ as $n, m \to \infty$.
Hence there exists a function $w_*(z) \in B = C^{-1}(D) \cap W^2_{p_0}(D)$, such that $S(w - w_*) \to 0$ as $n \to \infty$. It can be seen that $w_*(z)$ is a solution of Problem N for (3.5) with $t \in E$.

Similarly to the proof of Theorem 1.2, Chapter I, [7], from Problem N for (3.1) with $t = t_0 = 0$ is solvable, we may derived Problem N for (3.1) with $t = 1$ is solvable. In particular, Problem N for (3.1) with $A(z) = (1 - t)F(z, 0, 0, 0, 0), t = 1$, i.e. (3.1) is solvable. This completes the proof.

Now we estimate the difference of the solution of Problem N for (3.1) and its ap-proximations, and give the following result.

Theorem 3.2 *Suppose that the complex equation* (3.1) *satisfies the same conditions in Theorem* 3.1*. Then the difference* $w - w_n^t$ *of the solution* $w(z)$ *of Problem* N *for* (3.1) *and its approximative solution* $w_n^t = w_n(z, t)$ *possesses the following accuracy:*

$$S(w - w_n^t) = C^{-1}(R^0(w - w_n^t), \overline{D}] + L_{p_0}[RS(|(w - w_n^t)_{z\bar{z}}| + |(w - w_n^t)_{zz}|+$$
$$+ |(\overline{w - w_n^t})_{zz}|), \overline{D}] \le \gamma k\left[\frac{|1 - (\gamma|t - t_0|)^n}{1 - \gamma|t - t_0|}(1 - t) + (\gamma|t - t_0|)^n(1 - t_0)\right], \tag{3.13}$$

where $\gamma = M(q_1 + q_2 + k_0)$, $k = M(k_3 + k_5 + k_6)$, $0 \le t_0$, $t \le 1$, q_1, q_2, $k_j(j = 0, 3, 5, 6)$, M *are nonnegative constants as stated in Condition* C, (2.25) *and* (3.8).

Proof From (3.1) and (3.7) with $A(z) = (1 - t)F(z, 0, 0, 0, 0)$, we have

$$\begin{aligned}&(w - w^t_{n+1})_{z\bar z} = f(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}) - t_0 f(z, w^t_{n+1}, w^t_{n+1z}, \bar w^t_{n+1z}, \\ &w^t_{n+1zz}, \bar w^t_{n+1zz}) - (t - t_0) f(z, w^t_n, w^t_{nz}, \bar w^t_{nz}, w^t_{nzz}, \bar w^t_{nzz}) = \\ &= t_0 g(w, w^t_{n+1}) + (t - t_0) g(w, w^t_n) + (1 - t) f(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}),\end{aligned} \qquad (3.14)$$

where

$$f(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}) = F(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}) - F(z, 0, 0, 0, 0)$$
$$g(w, w^t_n) = f(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}) - f(z, w^t_n, w^t_{nz}, \bar w^t_{nz}, w^t_{nzz}, \bar w^t_{nzz}).$$

By (3.4) it is easy to see that

$$\begin{aligned}&g(w, w^t_n) = \tilde{Q}_1 (w - w^t_n)_{zz} + \tilde{Q}_1 (\bar w - \bar w^t_n)_{zz} + \tilde{A}_1 (w - w^t_n)_z \\ &+ \tilde{A}_2 (\bar w - \bar w^t_n)_{\bar z} + \tilde{A}_3 (w - w^t_n),\ |\tilde{Q}_j| \le q_j,\ j = 1, 2,\ q_1 + q_2 < 1, \\ &L_{p_0}[\tilde{A}_j, \bar D] \le k_{j-1} \le k_0,\ j = 1, 2, 3,\ p > 2,\end{aligned}$$

and then

$$\begin{aligned}&L_{p_0}[(t - t_0)RSg(w, w^t_n), \bar D] \le |t - t_0|[q_1 L_{p_0}(RS(w - w^t_n)_{zz}, \bar D) \\ &+ q_2 L_{p_0}(RS(\bar w - \bar w^t_n)_{zz}, \bar D) + k_0 C^1(R'(w - w^t_n), \bar D)] \le |t - t_0|(q_1 + q_2 + k_0) S(w - w^t_n, \bar D),\ L_{p_0} \\ &[(1 - t)RSf(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}), \bar D] \le (1 - t)[q_1 L_{p_0}(RSw_{zz}, \bar D) + \\ &+ q_2 L_{p_0}(RS\bar w_{zz}, \bar D) + k_0 C^1(R'w, \bar D)] \le (1 - t)(q_1 + q_2 + k_0) S(w),\end{aligned}$$

where

$$S(w) = C^1(R'w, \bar D) + L_{p_0}[RS(|w_{z\bar z}| + |w_{zz}| + |\bar w_{zz}|), \bar D],$$
$$C^1(R'w, \bar D) = C(Rw, \bar D) + C(Rw_z, \bar D) + C(R\bar w_z, \bar D).$$

In addition, the function $w(z) - w^t_{n+1}(z)$ satisfies the homogeneous boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda_1(z)}(w - w^t_{n+1})_z + \sigma(z)(w - w^t_{n+1})] = h_1(z), \\ &\mathrm{Re}[\overline{\lambda_2(z)}(w - w^t_{n+1})] = h_2(z),\ z \in \Gamma,\end{aligned} \qquad (3.15)$$

$$\begin{aligned}&\mathrm{Im}[\overline{\lambda_1(z)}(w - w^t_{n+1})_z + \sigma(z)(w - w^t_{n+1})]|_{z=a_j} = 0,\ j \in J_1, \\ &\mathrm{Im}[\overline{\lambda_2(z)}(w - w^t_{n+1})]|_{z=a_j} = 0,\ j \in J_2.\end{aligned} \qquad (3.16)$$

On the basis of Theorem 5.6 of Chapter 1, it can be obtained

$$\begin{aligned}&S(w - w^t_{n+1}) \le M(q_1 + q_2 + k_0)[|t - t_0| S(w - w^t_n) + (1 - t)S(w)] \\ &\le \gamma^{n+1}|t - t_0|^{n+1} S(w - w^t_0) + \gamma(1 - t)\frac{1 - \gamma^{n+1}|t - t_0|^{n+1}}{1 - \gamma|t - t_0|} S(w),\end{aligned} \qquad (3.17)$$

where $\gamma = M(q_1 + q_2 + k_0)$ and $w^t_0 = w(z, t_0)$ is the solution of Problem N for (3.7) with $t = t_0$ and $A(z) = (1 - t_0)F(z, 0, 0, 0, 0)$. Due to $w(z)$ is a solution of Problem N for (3.1), and $w - w^t_0$ is a solution of the following boundary value problem

$$(w - w^t_0)_{z\bar z} - t_0 g(w, w^t_0) = (1 - t) f(z, w, w_z, \bar w_z, w_{zz}, \bar w_{zz}), \qquad (3.18)$$

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda_1(z)}(w - w_0^t)_z + \sigma(z)(w - w_0^t)] = h_1(z),\\&\mathrm{Re}[\overline{\lambda_2(z)}(w - w_0^t)] = h_2(z),\ z \in \Gamma,\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\mathrm{Im}[\overline{\lambda_1(z)}(w - w_0^t)_z + \sigma(z)(w - w_0^t)]|_{z=a_j} = 0,\ j \in J_1,\\&\mathrm{Im}[\overline{\lambda_2(z)}(w - w_0^t)]|_{z=a_j} = 0,\ j \in J_2,\end{aligned} \tag{3.20}$$

we can conclude

$$S(w) \le M_9(k_3 + k_5 + k_6) = k, \tag{3.21}$$

$$\begin{aligned}S(w - w_0^t) &\le M(1 - t_0)L_{p_0}[RSf(z, w, w_z, \bar{w}_z, w_{zz}, \bar{w}_{zz}), \overline{D}]\\&\le M(q_1 + q_2 + k_0)(1 - t_0)S(w) \le \gamma(1 - t_0)k.\end{aligned} \tag{3.22}$$

Thus from (3.17), it follows that

$$\begin{aligned}S(w_{n+1} - w^t) &\le \gamma^{n+1}|t - t_0|^{n+1}\gamma(1 - t_0)k + \frac{\gamma(1 - t)k[1 - \gamma^{n+1}|t - t_0|^{n+1}]}{1 - \gamma|t - t_0|}\\&= \gamma k\left[\gamma^{n+1}|t - t_0|^{n+1}(1 - t_0) + \frac{(1 - t)(1 - \gamma^{n+1}|t - t_0|^{n+1})}{1 - \gamma|t - t_0|}\right].\end{aligned}$$

Hence (3.13) is true.

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