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Fuzzy Prime Ideals And Filters Of Lattices

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Abstract
A complete lattice (L, $\leq$) satisfying the infinite meet distributivity is called a frame. For a given bounded distributive lattice (X, $\wedge$, $\vee$) and a frame L, we introduce the notions of prime L-fuzzy ideals and L-fuzzy prime ideals of X and prove certain characterization theorems for these. Using the duality principle in lattices, the results on prime L-fuzzy ideals and L-fuzzy prime ideals are extended for filters also

Keywords: Lattices, complete lattices, frame, distributive lattice, ideals, filters, prime ideals, prime filters.

Introduction

The concept of prime fuzzy ideal was first introduced by U.M.Swamy and D.V.Raju [6] and later B.B.N.Koguep, C.N.Kuimi and C.Lele [3] discussed certain properties of prime fuzzy ideals of lattices when the truth values are taken from the interval [0, 1] of real numbers. After J.Goguen [2] and U.M.Swamy and others [5,6 and 7] have asserted that the interval [0, 1] is not sufficient to take the truth values general fuzzy statements, it was found that an abstract frame is most suitable to take the truth values. A complete lattice (L, $\leq$) satisfying the infinite meet distributivity, $a\wedge(supX) = sup\{a\wedge x \mid x \in X\}$ for any $a \in L$ and $X \subseteq L$, is called a frame. In this paper, we extended the results of the above works to the case when the truth values are taken from a general frame and obtain certain comprehensive results on primeness and irreducibility among L-fuzzy ideals of general lattices. Even though some of the results are true for general lattices, we concentrate on L-fuzzy ideals of bounded distributive lattices. We completely characterize the prime L-fuzzy ideals and filters and obtain a one-to-one correspondence between the prime L-fuzzy ideals of a distributive lattice X and the pairs (I, α) where I is a prime ideal of X and α is an irreducible element in the frame L. The discussion here mainly uses the transfert principle of M.Kondo and W.A.Dubek [4] in fuzzy theory.

The required basic concepts and results on the theory of partially ordered sets and lattices are refered to [1], while those on fuzzy sets to those mentioned in the references given at the end.

PRELIMINARIES

A partially ordered set $(X, \leq)$ is called a lattice (complete lattice) if every nonempty finite subset (respectively, every subset) of X has infimum and supremum in X. For any $A \subseteq X$, the iinfimum (supremum) of A, if they exist,

are denoted by $infA$ or $\bigwedge A$ or $\bigwedge_{a \in A} a$ ($supA$ or $\bigvee A$ or $\bigvee_{a \in A} a$ respectively). A lattice can also be described as an algebra $(X, \wedge, \vee)$ where $\wedge$ and $\vee$ are binary operations on X which are both associative, commutative and idempotent and satisfy the absorption laws $a \wedge (a \vee b) = a = a \vee (a \wedge b)$; in this case, the partial order $\leq$ on X is defined by $a \leq b \Leftrightarrow a = a \wedge b (\Leftrightarrow a \vee b = b)$ and $a \wedge b$ and $a \vee b$ are respectively the infimum and supremum of $\{a, b\}$ in X. A lattice $(X, \wedge, \vee)$ is said to be distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all a, b and c in X and is said to be bounded if there exists elements 0 and 1 in X such that $0 \leq a \leq 1$ for all $a \in X$. A nonempty subset I of a lattice $(X, \wedge, \vee)$ is said to be an ideal of X if, for any a and $b \in I$, $a \vee b \in I$ and $a \wedge x \in I$ for all $x \in X$. The set $I(X)$ of all ideals of a bounded lattice X forms a complete lattice under the set inclusion ordering. A complete lattice $(X, \leq)$ satisfying the infinite meet distributivity (that is, $x \wedge (supA) = sup\{x \wedge a \mid a \in A\}$ for all $x \in X$ and $A \subseteq X$) is called a frame.

Through out this paper X denotes a bounded distributive lattice $(X, \wedge, \vee)$ in which 0 and 1 are the smallest and greatest elements and L denotes a frame. Any mapping of X into L is called an L-fuzzy subset of X. An L-fuzzy subset A of X is called an L-fuzzy ideal of X if $A(0) = 1$ and $A(x \vee y) = A(x) \wedge A(y)$ for all x and $y \in X$. For any L-fuzzy subset A of X, the α-cut $A := \{x \in X \mid \alpha \leq A(x)\}$ is an ideal of X for all $\alpha \in L$ if and only if A is an L-fuzzy ideal of X. The set $FI_L(X)$ of all L-fuzzy ideals of X forms a complete lattice under the point-wise ordering. Since X is a distributive lattice, it follows that $FI_L(X)$ is also a distributive lattice. Also, $FI_L(X)$ is an algebraic fuzzy system, in the sense that it is closed under point-wise infimums and closed under point-wise supremums of direct above subclasses. Using these and the fact that L is frame, we get that $FI_L(X)$ satisfies the infinite meet distributivity and hence a frame.

PRIME *L*-FUZZY IDEALS

A proper ideal P in a lattice $(X, \wedge, \vee)$ is called prime if, for any a and $b \in X$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$ which is equivalent to saying that, for any ideals I and J of X, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. In general, an element p in a lattice X is called prime if p is not the largest element and, for any a and $b \in P$, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. This is to say that prime elements in the lattice $I(X)$ od ideals of X are precisely the prime ideals of X. In this section we discuss the prime elements in the lattice $FI_L(X)$ of all L-fuzzy ideals of X. Throughout this paper X denotes a bounded distributive lattice and L denotes a frame.

De nition 3.1. Let $X = (X, \wedge, \vee)$ be a bonded distributive lattice and L a frame. An L-fuzzy ideal A of X is called proper if A is not the constant map 1; that is, $A(X) \neq 1$ for some $x \in X$. A proper L-fuzzy ideal P of X is called a prime L-fuzzy ideal of X if for any L-fuzzy ideals A and B of X,

$A \wedge B \leq P \Longrightarrow A \leq P$ or $B \leq P$.

Since X is a distributive lattice, so are the lattice $I(X)$ of ideals of X and the lattice $FI_L(X)$ of L-fuzzy ideals of X. Therefore a proper L-fuzzy ideal P of X is prime if and only if it is irreducible in the sense that, for any L-fuzzy ideals A and B of X,

$P = A \wedge B \Longrightarrow P = A$ or $P = B$.

De nition 3.2. For any ideal I of X and $\alpha \in L$, define $\alpha_I : X \longrightarrow L$ by

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1, if $x \in I$

$\alpha_I(x) =$
α, if $x \notin I$.

Then α_I is an L-fuzzy ideal of X and is called α-level L-fuzzy ideal corre-sponding to I.

For any fixed $\alpha \neq 1$ in L, I α_I gives us an embedding of I(X) into $(FI)_L(X)$. The following is a method of constructing prime L-fuzzy ideals.

Theorem 3.3. *Let $X = (X, \wedge, \vee)$ be a bounded distributive lattice and L a frame. Let I be an ideal of X and $\alpha \in L$. Then the α-level L-fuzzy ideal α_I is a prime L-fuzzy ideal of X if and only if I is a prime ideal of X and α is a prime element in L.*
we have $\alpha_J \wedge \alpha_K = \alpha_{J\cap K}$ and $\alpha_J \leq \alpha_K$ if and only $J \subseteq K$. From these, it follows that I is a prime ideal of X. For β and $\gamma \in L$, we have

$\beta \wedge \gamma \leq \alpha \Longrightarrow (\beta \wedge \gamma)_I \leq \alpha_I \Longrightarrow \beta_I \wedge \gamma_I \leq \alpha_I$

$\Longrightarrow \beta_I \leq \alpha_I$ or $\gamma_I \leq \alpha_I \Longrightarrow \beta \leq \alpha$ or $\gamma \leq \alpha$.
Therefore α is a prime element in L.

Conversely suppose that $I$is a prime ideal of X and α is a prime element in L. Then $I \neq X$ and $\alpha < 1$ and hence α_I is a proper L-fuzzy ideal of X. Let A and B be any L-fuzzy ideals of X. Let A and B be any L-fuzzy ideals of X such that $A \wedge B \leq \alpha_I$. Suppose that A α_I . Then there exists $x \in X$ such that $A(x)$ $\alpha_I(x)$. Therefore $\alpha_I(x) \neq 1$ and hence $\alpha_I(x) = \alpha$ and $x \notin I$. Now, for any $y \notin I$, $x \wedge y \notin I$ (since I is prime) and
$A(x \wedge y) \wedge B(x \wedge y) = (A \wedge B)(x \wedge y) \leq \alpha_I(x \wedge y) = \alpha$.

Since α is prime, $A(x\wedge y) \leq \alpha$ or $B(x\wedge y) \leq \alpha$. But, since $A(x)$ $\alpha_I(x) = \alpha$ and $A(x) \leq A(x \wedge y)$, we have $A(x \wedge y)$ α. Therefore $B(x \wedge y) \leq \alpha$. Since $B(y) \leq B(x \wedge y)$, we get that $B(y) \leq \alpha$. Therefore $B(y) \leq \alpha_I(y)$ for all $y \notin I$ and hence $B \leq \alpha_I$. Thus α_I is a prime L-fuzzy ideal of X.

The above theorem together with the following leads to a characteriza-tion of prime L-fuzzy ideals.

Theorem 3.4. *Let P be a proper L-fuzzy ideal of a bounded distributive lat-tice X. Then P is prime if and only if the following conditions are satis ed.*

P assumes exactly two values

$P(1)$ is a prime element in L

$\{x \in X \mid P(x) = 1\}$ is a prime ideal of X.

Proof. Suppose that P satisfies the conditions (1), (2) and (3). Since P is antitone, we have $P(1) \leq P(x) \leq P(0) = 1$ for all $x \in X$. Let $P(1) = \alpha$ and $I = \{x \in X \mid P(x) = 1\}$. Then I is a prime ideal of X and α is a prime element in L. Also, by (1), α and 1 are the only values of P. Therefore $P = \alpha_I$ and hence, by the above theorem, P is a prime L-fuzzy ideal of X.

Conversely suppose that P is a prime L-fuzzy ideal of X. Then $P(1) \leq P(x) \leq P(0) = 1$ for all $x \in X$. Let $P(1) = \alpha$ and $I = \{x \in X \mid P(x) = 1\}$. Since P is proper, we get that $\alpha < 1$. Let $x \in X$ and $\beta = P(x)$. Then $\alpha \leq \beta \leq 1$. Put $J = \{y \in X \mid \beta \leq P(y)\}$. Then I and J are ideals of X and
$\beta_I \wedge \alpha_J \leq P$.

Since P is prime, $\beta_I \leq P$ or $\alpha_J \leq P$. If $\beta_I \leq P$, then $\beta = \beta_I\,(1) \leq P\,(1) = \alpha$ and hence $\beta = \alpha$. If $\alpha_J \leq P$, then

$1 = \alpha_J\,(x) \leq P\,(x) = \beta$ and hence $\beta = 1$.

Thus P assumes exactly two values, namely 1 and α. Now, since P is prime and $P = \alpha_I$, it follows from the above theorem, that I is a prime ideal of X and α is a prime element in L. □

Corollary 3.5. *P is a prime L-fuzzy ideal of X if and only if P is of the form α_I for some ideal I of X and prime element α in L.*

Corollary 3.6. $(\alpha, I) \rightarrow \alpha_I$ *is a one-to-one correspondence between the pairs* (α, I) *where α is a prime element in L and I is a prime ideal of X and the prime L-fuzzy ideals of X.*

L-FUZZY PRIME IDEALS

For any prime L-fuzzy ideal P of a bounded distributive lattice X, each β-cut of P, $\beta \in L$, is either a prime ideal of X or the whole of X. Infact, if $P = \alpha_I$ where α is a prime element of L and I is a prime ideal of X (refer corollary 3.5), the β-cut of P is given by $P = I$ or X depending on $\beta\ \alpha$ or $\beta \leq \alpha$ respectively. In the following we characterize the L-fuzzy ideals whose α-cuts are either prime ideals of X or the whole of X.

Theorem 4.1. *Let $(X, \wedge, \vee)$ be a bounded distributive lattice and L a frame. The following are equivalent to each other for any L-fuzzy ideal A of X.*

For any $\alpha \in L$, either $A\ = X$ or A is a prime ideal of X

For any $\alpha \in L$ and x and $y \in X$, $x \wedge y \in A \Rightarrow x \in A$ or $y \in A$

For any x and $y \in X$, $A(x \wedge y) = A(x)$ or $A(y)$

For any x and $y \in X$, $A(x \wedge y) = A(x) \vee A(y)$ and either $A(x) \leq A(y)$ or $A(y) \leq A(x)$.

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3) : Let x and $y \in X$. Put $\alpha = A(x \wedge y)$ Then $x \wedge y \in A$. By (2), $x \in A$ or $y \in A$. If $x \in A$, then

$A(x \wedge y) = \alpha \leq A(x) \leq A(x \wedge y)$ and hence $A(x \wedge y) = A(x)$.
If $y \in A$, then $A(x \wedge y) = A(y)$.

(3) $\Longrightarrow$ (4) : This is a consequence of the fact that
$A(x) \leq A(x \wedge y)$ and $A(y) \leq A(x \wedge y)$ for all x and $y \in X$.
(4) $\Longrightarrow$ (1) : Let $\alpha \in L$ such that $A \neq X$. Then A is a proper ideal of X. Let x and $y \in X$ such that $x \wedge y \in A$. Then

$\alpha \leq A(x \wedge y) = A(x) \vee A(y) = A(x)$ or $A(y)$ (by (4))

and hence $x \in A$ or $y \in A$. Therefore A is a prime ideal of X.

De nition 4.2. A proper L-fuzzy ideal A of X is called an L-fuzzy prime ideal of X if it satisfies one (and hence all) of the conditions in Theorem 4.1.

De nition 4.3. Let I be an ideal of a bounded distributive lattice X and α and β elements of a frame L. Define an L-fuzzy subset $(\alpha, \beta)_I$ of X by

$$(\alpha, \beta)_I(x) = \begin{cases} 1, & \text{if } x = 0 \\ \alpha, & \text{if } 0 \neq x \in I \\ \beta, & \text{if } x \notin I \end{cases}$$

Note that $(1, \beta)_I = \beta_I$ and $(1, 0)_I = \chi_I$, the characteristic map of I. In the following two results, whose proof are easy verifications, we observe that the prime ideals of X can be identified with L-fuzzy prime ideals of X.

Theorem 4.4. *The following are equivalent to each for any proper ideal I of X.*

I is a prime ideal of X

$(1, \beta)_I$ *is an L-fuzzy prime ideal of X for any* $\beta < 1$ *in L*

The characteristic map χ_I $(= (1, 0)_I)$ *is an L-fuzzy prime ideal of X.*

Theorem 4.5. *Suppose that I is a proper ideal of the lattice X and the least element* 0 *in X is prime. Then I is a prime ideal of X if and only if* $(\alpha, \beta)_I$ *is an L-fuzzy prime ideal of X for all* $1 \neq \beta \leq \alpha \in L$.

It can be easily verified that any prime L-fuzzy ideal of X is an L-fuzzy prime ideal of X and that the converse is not true. In the following we describe a method for constructing L-fuzzy prime ideals.

Theorem 4.6. *Let X be a bounded distributive lattice and C be a chain in a frame L such that* $1 \in C$ *and* $supD \in C$ *for all* $D \subseteq C$. *Let* $\{I\}_{\in C}$ *be a class of ideals of X such that* $\cap I = X$ *or I is a prime ideal of X for each*
$\alpha \in C$ *and, for any* $D \subseteq C$, $I = I_{supD:}$ *De ne an L-fuzzy subset P of* $\in D$
X by
$P(x) = sup\{\alpha \in C \mid x \in I\}$

for any $x \in X$. *Then P is an L-fuzzy prime ideal of X if P is proper. Con-versely, every L-fuzzy prime ideal of X is obtained by the above procedur*

$\Longrightarrow x \in I$

$\Longrightarrow I =$

Proof. First note that $0 = sup\phi \in C$. Also, by hypothesis, for any x and $y \in X$,

$x \wedge y \in I \iff x \in I$ or $y \in I$

and therefore we have

$P(x) \wedge P(y) = sup\{\alpha \in C \mid x \in I\} \wedge sup\{\beta \in C \mid y \in I\}$

$= sup\{\alpha \wedge \beta \mid \alpha, \beta \in C, x \in I, y \in I\}$ (by the infinite meet distributivity in L) $= sup\{\gamma \in C \mid x \vee y \in I\} = P(x \vee y)$ (since $I \cup I \subseteq I_\wedge$).

Therefore P is an L-fuzzy ideal of X. Also, clearly $I \subseteq P$, the α-cut of P, for each $\alpha \in C$. On the other hand

$x \in P \implies \alpha \leq P(x) = sup\{\beta \in C \mid x \in I\}$

$\implies \alpha = \alpha \wedge P(x) = sup\{\alpha \wedge \beta \mid \beta \in C, x \in I\}$
$\cap$
$\{I_{\wedge} \mid \beta \in C, x \in I\}$

(since $I \subseteq I_{\wedge}$).

Also, we have $x \wedge y \in I \Leftrightarrow x \in I$ or $y \in I$ for any $\alpha \in C$ and x and $y \in X$. From this, it follows that

$P(x \wedge y) = P(x) \vee P(y)$ and $P(x) \leq P(y)$ or $P(y) \leq P(x)$

since $P(x)$ and $P(y)$ are in the chain C. Thus, if P is proper, then P is an L-fuzzy ideal of X. For the converse, one can take $C = \{P(x) \mid x \in X\}$. □

In the above, note that P is proper if and only if I is a proper ideal of X for some $\alpha \in C$.

PRIMENESS IN L-FUZZY FILTERS

It is well known that by interchanging the operations $\wedge$ and $\vee$ in a lattice $(X, \wedge, \vee)$ we get another lattice $(X, \vee, \wedge)$ which is called the dual of X and is denoted by X^d; The partial orders in X and X^d are inverses to each other. The ideals of X^d are known as filters of X. In other words, a nonempty subset F of a lattice $X = (X, \wedge, \vee)$ is called a filter of X if a and
$b \in F \Rightarrow a \wedge b \in F$ and $a \vee x \in F$ for all $x \in X$.
De nition 5.1. An L-fuzzy subset F of a lattice $(X, \wedge, \vee)$ is called an
L-fuzzy filter of X if $F(1) = 1$ and

$F(x \wedge y) = F(x) \wedge F(y)$ for all x and $y \in X$.

Clearly every L-fuzzy filter of X is an isotone, in the sense that $x \leq y$ in $X \Rightarrow F(x) \leq F(y)$ in L. The set $FF_L(X)$ of all L-fuzzy filters of any lattice is a lattice under point-wise ordering and $FF_L(X)$ is a complete lattice if and only if X has greatest element. Also, $FF_L(X)$ is distributive if and only if so is X. Recall that the prime ideals of the dual lattice X^d are called prime filters of a lattice X.

De nition 5.2. An L-fuzzy filter F of X is said to be proper if $F(x) \neq 1$ for some $x \in X$. Let P be a proper L-fuzzy filter of a lattice X.

(1) P is said to be prime L-fuzzy filter of X if, for any L-fuzzy filters F and G of X,

$F \wedge G \leq P \implies F \leq P$ or $G \leq P$

P is said to be L-fuzzy prime filter of X if $P(x \vee y) = P(x)$ or $P(y)$ for any x and $y \in X$.

It is clear that, when X is a bounded distributive lattice, a proper L-fuzzy filter P of X is prime if and only if, for any L-fuzzy filters F and G of X, $P = F \wedge G \implies P = F or P = G$. All the results proved for prime L-fuzzy ideals and L-fuzzy prime ideals in the previous two sections hold good for filters also, since filters of X are simply the ideals of the dual lattice X^d. In particular, we have the following two characterization theorems.

Theorem 5.3. *Let* $(X, \wedge, \vee)$ *be a bounded distributive lattice and L a frame. A proper L-fuzzy lter P of X is prime if and only if there exist a unique prime lter F of X and a unique prime element* α *in L such that* $P = \alpha_F$
;

that is,
{
1, *if* $x \in F$
$P(x) =$
α, *if* $x \notin F$.

Theorem 5.4. *Let P be a proper L-fuzzy lter of a bounded distributive lattice X. Then P is an L-fuzzy prime lter of X if and only if, for any* $\alpha \in L$, *the* α*-cut P is either a prime lter of X or the whole of* The Authors thank Prof. U.M.Swamy for his help in preparing this paper.

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Proof. Suppose that α_I is a prime L-fuzzy ideal of X. Then α_I is a proper and hence I is a proper ideal of X and $\alpha \neq 1$. For any ideals J and K of X