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On The Conjugate Secondary Eigenvalues and Secondary Singular Values of A Complex Square Matrix

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Abstract:

In this paper, the conjugate secondary eigen values (con-s-eigen values) of a matrix, when properly defined, obey relations similar to the classical inequalities between the s-eigen values and s-singular values. Several interesting secondary spectral properties of conjugate secondary normal (con-s-normal) matrices are indicated. This matrix class plays the same role in the theory of s-unitary congruence as the class of s-normal matrices plays in the theory of s-unitary similarities.

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1. Introduction

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{n\times n}$, let A^T, $\overline{A}$, A^*, A^s, $A^\theta \left(= \overline{A}^s \right)$ and A^{-1} denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix A respectively. The conjugate secondary transpose of A satisfies the following properties such as $\left(A^\theta\right)^\theta = A$, $\left(A+B\right)^\theta = A^\theta + B^\theta$, $\left(AB\right)^\theta = B^\theta A^\theta$. etc

Definition 1

A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A.$

Definition 2

A Matrix $A \in C_{n\times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^*A}.$

Definition 3

A matrix $A \in C_{n\times n}$ is said to be secondary normal (s-normal) if $AA^\theta = A^\theta A.$

Definition 4

A matrix $A \in C_{n\times n}$ is said to be unitary if $AA^* = A^*A = I.$

Definition 5

A matrix $A \in C_{n\times n}$ is said to be *s*-unitary if $AA^{\theta} = A^{\theta}A = I$.

Definition 6

The spectrum of a matrix $A \in C_{n\times n}$ is the set of all eigen values of *A*.

Definition 7

The spectral radius of *A* is defined by $\rho(A) = \max\{|\lambda| / \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of *A*.

Definition 8

Matrices $A, B \in M_n(C)$ are said to be con-s-similar if $A = SB\overline{S}^{-1}$ for a non s-singular matrix $S \in M_n(C)$. As usual, the bar over the symbol of a matrix means element wise conjugation. s-unitary congruence is an important particular case of con-s-similarity where $S = U$ is an s-unitary matrix and $A = UBU^S$.

Definition 9

Let a scalar $\mu \in C$ and a nonzero vector $x \in C^n$ are called a con-s-eigen value and a con-s-eigen vector (associated with μ) of a matrix *A*, respectively, if

$$Ax = \mu\overline{x} \quad \text{. . . (1)}$$

Result 1

It follows from **[Sec. 4.6 of 1]** that μ is a con-s-eigen value of *A* if and only if $|\mu|^2$ is an s-eigen value of $\overline{A}A$. Therefore, if $\overline{A}A$ has no real nonnegative s-eigen values, then *A* has no con-s-eigen values. If μ is a con-s-eigen value, then, for all $\theta \in R, e^{i\theta}\mu$ also is a con-s-eigen value.

Hence if *A* has a con-s-eigen value, then it has infinitely many of them. By contrast, a matrix of order *n* always has exactly *n* s-eigen values if their multiplicities are counted. It follows that the set of con-s-eigen values is inconvenient to work with.

In **Sec. 2** of this paper, we suggest a different definition of con-s-eigen values. In accordance with this definition, any matrix of order *n* has exactly *n* con-s-eigen values (with account for their multiplicities). It turns out that certain relations between the (ordinary) s-eigen values and matrix norms and also between the s-eigen values and the s-singular values have counterparts for the con-s-eigen values.

Some classical inequalities, such as the schur inequality or the additive Weyl inequalities, become equalities for a s-normal matrix *A*. In **Sec. 3**, we show that in the case of con-s-eigenvalues, similar equalities hold for the con-s-normal matrices. In the theory of s-unitary congruences, this matrix class plays a role similar to that of the s-normal matrices in the theory of s-unitary similarities. Other analogous properties of matrices in these two classes are also indicated.

2. Inequalities between the Con-s-Eigen Values and the s-Singular Values

Given a matrix $A \in M_n(C)$, we associate with it the matrices

$$A_L = \bar{A}A \qquad \text{. . . (2)}$$

and

$$A_R = A\bar{A} \qquad \text{. . . (3)}$$

Although, in general, the products *AB* and *BA* need not be similar, the matrices A_L and A_R always are similar **[1, Sec. 4.6]**. Therefore, in the subsequent discussion of secondary spectral properties of these matrices, it will be sufficient to consider only one of them, say, A_L .

The secondary spectrum of A_L has the following remarkable properties.

1. It is s-symmetric about the real axis. Moreover, the s-eigen values λ and $\bar{\lambda}$ are of the same multiplicity.
2. The negative real s-eigen values of A_L (if any) are necessarily of even algebraic multiplicity.

Let $\lambda_S(A_L) = \{\lambda_1, ..., \lambda_n\}$ **. . . (4)**

be the secondary spectrum of A_L and let $\rho_S(A) = \max\{|\lambda|, \lambda \in \lambda_S(A)\}$ denote the secondary spectral radius of *A*.

Definition 10

The con-s-eigen values of *A* are the *n* scalars $\mu_1, ..., \mu_n$ defined as follows:

- If $\lambda_i \in \lambda_S(A_L)$ does not lie on the negative real semi-axis, then the corresponding con-s-eigen value μ_i is defined as the square root of λ_i with nonnegative real part, and the multiplicity of μ_i is that of λ_i

  i.e, $\mu_i = \lambda_i^{\frac{1}{2}}, \operatorname{Re}\mu_i \geq 0$ **. . . (5)**

- With a real negative s-eigen value $\lambda_i \in \lambda_S(A_L)$ we associate two conjugate purely imaginary con-s-eigen values

  $\mu_i = \pm\lambda_i^{\frac{1}{2}}$ **. . . (6)** The multiplicity of each of them being half the multiplicity of λ_i.

The set $C\lambda_S(A) = \{\mu_1, ..., \mu_n\}$ **. . . (7)**

is called the conjugate secondary spectrum of *A*.

The con-s-eigen values of a matrix *A* allow for another interpretation. Define the matrix

$$\hat{A} = \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix} \qquad \text{. . . (8)}$$

Proposition 1

Let $\mu_1,\ldots,\mu_n$ be the con-s-eigen values of a $n\times n$ matrix *A*. Then

$$\lambda_S\left(\hat{A}\right)=\left\{\mu_1,\ldots,\mu_n,-\mu_1,\ldots,-\mu_n\right\} \quad \textbf{. . . (9)}$$

Proof

The assertion desired follows from two observations. First, we have $\hat{A}^2=A_R\oplus A_L$, which implies that any s-eigen value of $\hat{A}$ is a square root of an s-eigen value of A_L. Second, the characteristic polynomial $\varphi(\lambda)$ of $\hat{A}$ is given by

$$\varphi(\lambda)=\det\left(\lambda I_{2n}-\hat{A}\right)=\det\left(\lambda^2 I_n-A_L\right)=\det\left(\lambda^2 I_n-A_R\right)$$

Thus, if λ is an s-eigen value of $\hat{A}$, then $-\lambda$ also is an s-eigen value of $\hat{A}$, and both of them have the same multiplicity.

For the rest of this section, we adopt the following conventions.

(i) The s-singular values of *A* are arranged in non increasing order,

$$i.e.,\sigma_1(A)\geq\sigma_2(A)\geq\ldots\geq\sigma_n(A) \quad \textbf{. . . (10)}$$

Hence $\quad i.e.,\sigma_{\max}(A)=\sigma_1(A)=\|A\|_2 \quad$ **. . . (11)**

(ii) The con-s-eigen values of *A* are numbered in non increasing order of their absolute values,

i.e., $|\mu_1(A)|\geq|\mu_2(A)|\geq\ldots\geq|\mu_n(A)|$ **. . . (12)**

(iii)The same conventions apply to the s-singular values and s-eigen values of $\hat{A}$,

$$i.e.,\sigma_1\left(\hat{A}\right)\geq\sigma_1\left(\hat{A}\right)\geq\ldots\geq\sigma_{2n}\left(\hat{A}\right) \quad \textbf{. . . (13)}$$

$$\left|\lambda_1\left(\hat{A}\right)\right|\geq\left|\lambda_2\left(\hat{A}\right)\right|\geq\ldots\geq\left|\lambda_{2n}\left(\hat{A}\right)\right| \quad \textbf{. . . (14)}$$

In view of **(9)**, we have

$$\left|\lambda_{2i-1}\left(\hat{A}\right)\right|=\left|\lambda_2\left(\hat{A}\right)\right|=|\mu_i(A)|, i=1,2,\ldots,n. \quad \textbf{. . . (15)}$$

Note that $\hat{A}$ has the same s-singular values as $A\oplus\overline{A}$ and $\overline{A}$ has the same s-singular values as *A*. Consequently, the s-singular values of $\hat{A}$ are those of *A* repeated twice.

Thus, $\sigma_{2i-1}\left(\hat{A}\right)=\sigma_{2i}\left(\hat{A}\right)=\sigma_i(A), i=1,2,\ldots,n.$ **. . . (16)**

Finally, we define the conjugate secondary spectral radius of *A* as follows:

$$C\rho_S(A)=|\mu_1(A)| \quad \textbf{. . . (17)}$$

Proposition 2

Let $\|\cdot\|$ be an absolute matrix norm. Then

$$C\rho_S(A) \le \|A\| \qquad \text{. . . (18)}$$

Proof

We have $C\rho_s^2(A) = |\mu_1(A)|^2 = \rho_S(\bar{A}A) \le \|\bar{A}A\| \le \|\bar{A}\|\|A\| = \|A\|^2$.

The secondary spectral norm is not absolute. However, inequality **(18)** holds true for the secondary spectral norm as well.

Proposition 3

The following inequality is valid.

$$C\rho_S(A) \le \|A\|_2 \qquad \text{. . . (19)}$$

Proof

For any sub multiplicative matrix norm $\|\cdot\|$, we have $\rho_S(\hat{A}) \le \|\hat{A}\|$, implying that $|\lambda_1(\hat{A})| \le \sigma_1(\hat{A})$

In view of **(11)** and **(15)-(17)** this is the desired inequality **(19)** in disguised form.

Remark 1

In a personal communication, R. Horn, indicated to the author that **Propositions 2** and **3** can be united and strengthened under the assumption that for the matrix norm used, $\|A\| = \|\bar{A}\|$. Indeed, in this more general case, the proof of **Proposition 2** remains the same. In particular, not only the secondary spectral norm but all the s-unitarily invariant norms are covered.

Proposition 4

The con-s-eigen values satisfy the inequality

$$\sum_{i=1}^{n} |\mu_i(A)|^2 \le \|A\|_F^2 \qquad \text{. . . (20)}$$

Proof

In application to $\hat{A}$, the well-known schur inequality yields

$$\sum_{i=1}^{2n} |\lambda_i(\hat{A})|^2 \le \|\bar{A}\|_F^2 \qquad \text{. . . (21)}$$

Obviously, $\|\hat{A}\|_F^2 = 2\|A\|_F^2$ which, together with **(15)**, shows that **(21)** is equivalent to **(20)**.

Since, $\|A\|_F^2 = \sum_{i=1}^{n} \sigma_i^2(A)$, relation **(20)** can be regarded as an inequality between the con-s-eigen values of A and its s-singular values. From this point of view, the following theorem is an extension of **Proposition 4**.

Theorem 1

For $1 \le m \le n$ and an arbitrary real non negative μ,

$$\sum_{i=1}^{m} |\mu_i(A)|^{\delta} \le \sum_{i=1}^{m} \sigma_i^{\delta}(A) \qquad \textbf{. . . (22)}$$

Proof

By applying the additive Weyl inequalities **[3, sec.II.4.2]** to $\hat{A}$, we obtain

$$\sum_{i=1}^{l} \left|\lambda_i\left(\hat{A}\right)\right|^{\mu} \le \sum_{i=1}^{l} \sigma_i^{\mu}\left(\hat{A}\right), 1 \le l \le 2n \qquad \textbf{. . . (23)}$$

Setting $l = 2m(1 \le m \le n)$ in **(23)** and taking into account **(15)** and **(16)**, we arrive at **(22)**.

Proposition 5

Let A be a block triangular matrix of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$.

Then $$C\lambda_S(A) = C\lambda_S(A_{11}) \cup C\lambda_S(A_{22}) \qquad \textbf{. . . (24)}$$

Of course, **(24)** also holds for a lower block triangular matrix. Moreover, analogous equalities are valid not only for 2×2 block triangular matrices but for all block orders.

3. Con-s-Normal Matrices

The role of s-normal matrices in the theory of s-unitary similarities is well known. It is related to the fact that the s-normal matrices are exactly the matrices that can be brought to the simplest (secondary diagonal) form by s-unitary similarity transformations. The con-s-normal matrices (c.s.n. matrices) play a similar role in the theory of s-unitary congruences.

Definition 11 [2]

A matrix $A \in C_{n \times n}$ is said to be a conjugate secondary normal matrix (con-*s*-normal) if $AA^{\theta} = \overline{A^{\theta} A}$ where $A^{\theta} = \overline{A}^{S}$. $\qquad$ **. . . (25)**

Theorem 2

Any con-s-normal matrix $A \in M_n(C)$ can be brought by a proper s-unitary congruence transformation to a block diagonal matrix with diagonal blocks of order 1 and 2. The 1×1 blocks are the nonnegative con-s-eigen values of *A*. Each 2×2 block corresponds to a pair of complex con con-s-eigen values $\mu_j = \rho_j e^{i\theta_j}, \bar{\mu}_j$ and is of the form $\begin{bmatrix} 0 & \rho_j \\ \rho_j e^{-i2\theta_j} & 0 \end{bmatrix}$

. . . (26)

or $\begin{bmatrix} 0 & \mu_j \\ \bar{\mu}_j & 0 \end{bmatrix}$ **. . . (27)**

The block diagonal matrix described in **Theorem 2** is called the canonical form of the con-s-normal matrix *A*. The form **(26)** of its 2×2 blocks was used in **[4]**, whereas the alternative form **(27)** was given in **[5]**.

Complex s-symmetric, s-skew symmetric and s-unitary matrices are special cases of con-s-normal matrices. From the classical Takagi theorem **[1, sec.4.4]** it follows that the con-s-eigen values of a s-symmetric matrix are identical to its s-singular values. The con-s-eigen values of a s-unitary matrix U, being the square roots of the s-eigen values of the s-unitary matrix $\bar{U}U$, have unit absolute values; on the other hand, all the s-singular values of *U* are equal to one. This relation between the con-s-eigen values and the s-singular values holds for the entire class of con-s-normal matrices.

Proposition 6

The s-singular values of a con-s-normal matrix *A* are the absolute values of its con-s-eigen values.

Proof

The relation desired is readily obtained by inspecting the canonical form of *A*. Indeed, the nonnegative con-s-eigen values (i.e., 1×1 the blocks in the canonical form) are s-singular values of *A*. on the other hand, the s-singular spectrum of matrix **(27)** is the scalar $|\mu_j|$ repeated twice.

Corollary 1

For a con-s-normal matrix *A*, inequalities **(19), (20)** and **(22)** hold with equality.

Remark 2

The Toeplitz (or Cartesian) decomposition of a complex square matrix *A* is defined as the representation,

$$A = B + C, \quad B = B^{\theta}, \quad C = -C^{\theta}$$ **. . . (28)**

The matrices *B* and *C*, called the real and imaginary parts of *A*, respectively are uniquely determined by the equalities

$$B = \frac{1}{2}\left(A + A^{\theta}\right), \ C = \frac{1}{2}\left(A - A^{\theta}\right).$$

The usefulness of the Toeplitz decomposition is related to the fact that it is respected by s-unitary similarity transformations in the following sense: for a s-unitary matrix U, the matrices $U^{\theta}BU$ and $U^{\theta}CU$ are the real and imaginary parts of $U^{\theta}AU$ respectively; in addition, under the transformation with the matrix U, all the three matrices *A*, *B* and *C* preserve their s-eigen values.

The representation $A = S + K$ **. . . (29)**

of a matrix *A*, where

$$S = \frac{1}{2}\left(A + A^{S}\right) \text{ and } K = \frac{1}{2}\left(A - A^{S}\right) \quad \textbf{. . . (30)}$$

are a s-symmetric and a s-skew symmetric matrices, called the s-symmetric and s-skew symmetric parts of *A*, respectively, will be referred to as its SSSSS (meaning s-Symmetric-s-Skew Symmetric) decomposition. Decomposition **(29), (30)** is the counterpart of the Toeplitz decomposition for the theory of s-unitary congruences.

Decomposition **(29), (30)** is respected by s-unitary congruence transformation in the sense that for a s-unitary U, the matrices $U^{S}SU$ and $U^{S}KU$ are the s-symmetric and s-skew symmetric parts of the matrix $U^{S}AU$ respectively. Moreover the con-s-eigen values of the three matrices *A*, *S* and *K* are preserved under s-unitary congruence transformations.

Theorem 3

Let *A* be a con-s-normal matrix with SSSSS decomposition **(29), (30)**. Then the con-s-eigen values of the matrices *S* and *K* are the real and imaginary parts, respectively, of the con-s-eigen values of *A*.

Proof

This can readily be seen by inspecting the canonical form of *A*. If μ is a 1×1 block in the canonical form, then, obviously, its SSSSS decomposition is $\mu = \mu + 0$.

If $\mu_j = x_j + iy_j$ is a complex con-s-eigen value of *A*, then the SS SSS decomposition of matrix **(27)** is of the form $S_j + K_j$,

where $$S_j = \begin{bmatrix} 0 & x_j \\ x_j & 0 \end{bmatrix} \quad \textbf{. . . (31)}$$

and $$K_j = \begin{bmatrix} 0 & iy_j \\ -iy_j & 0 \end{bmatrix} \quad \textbf{. . . (32)}$$

The conjugate secondary spectrum of matrix **(31)** is the scalar x_j repeated twice, whereas matrix **(32)** has the con-s-eigen values iy_j and $-iy_j$.

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