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Vertex-Edge Dominating Sets and Vertex-Edge Domination Polynomials of Cycles

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Abstract

Let G = (V, E) be a simple Graph. A set $S \subseteq V(G)$ is a vertex-edge dominating set (or simply ve-dominating set) if for all edges $e \in E(G)$, there exist a vertex $v \in S$ such that v dominates e. In this paper, we study the concept of vertex-edge domination polynomial of the cycle C_n. The vertex-edge domination polynomial of C_n is $D_{ve}(C_n, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{|V(G)|} d_{ve}(C_n, i)x^i$, where $d_{ve}(C_n, i)$ is the number of vertex-edge dominating sets of C_n with cardinality i. We obtain some properties of $D_{ve}(C_n, x)$ and its co-efficients. Also, we calculate the recursive formula to derive the vertex-edge domination polynomials of cycles.

Keywords: Cycle, vertex-edge dominating sets, vertex-edge domination polynomial, vertex-edge domination number.

1. Introduction

Let G = (V, E) be a simple graph of order |V| = n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \cup_{v\in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if N[S] = V, or equivalently, every vertex in V\S is adjacent to atleast one vertex in S. The domination number of a graph, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. A set of vertices in a Graph G is said to be a vertex-edge dominating set, if for all edges $e \in E(G)$ there exists a vertex $v \in S$ such that v dominates e. Otherwise for a graph G = (V, E), a vertex $u \in V(G)$ ve-dominates an edge $vw \in E(G)$ if (i) u = v or u = w (u is incident to vw) or (ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

The minimum cardinality of a ve-dominating set of G is called the vertex-edge domination number of G, and is denoted by $\gamma_{ve}(G)$. A cycle can be defined as a closed path, and is denoted by C_n.

In the next section we construct the families of the vertex-edge dominating sets of cycles by recursive method. In section 3, we use the results obtained in section 2 to study the vertex-edge domination polynominal of cycles.

We use the notation $\lceil x \rceil$, for the smallest integer greater than or equal to *x*; also we denote the set {1, 2, . . . , n} by [n], throughout this paper.

2. Vertex-edge dominating sets of cycles

Let $D_{ve}(C_{n,i})$ be the family of vertex-edge dominating sets of C_n with cardinality i.

Lemma: 2.1 $\gamma_{ve}(C_n) = \left\lceil \frac{n}{4} \right\rceil$

Lemma: 2.2 $D_{ve}(C_j, i) = \phi$, iff $i > j$ or $i < \left\lceil \frac{j}{4} \right\rceil$.

Proof:

As C_j contains j vertices, any member of $D_{ve}(C_j, i)$ contains atmost j vertices.
Therefore, $D_{ve}(C_j, i) = \phi$ for $i > j$.

Also, since $\left\lceil \frac{j}{4} \right\rceil$ is minimum cardinality of a vertex-edge dominating set, there is no vertex-edge dominating set of cardinality less than $\left\lceil \frac{j}{4} \right\rceil$. Therefore, $D_{ve}(C_j, i) = \phi$ for $i < \left\lceil \frac{j}{4} \right\rceil$.

Hence $D_{ve}(C_j, i) = \phi$ if $i > j$ or $i < \left\lceil \frac{j}{4} \right\rceil$.

Conversly,

If $i > j$ or $i < \left\lceil \frac{j}{4} \right\rceil$, then

$d_{ve}(C_{j,i}) = 0$

$\therefore D_{ve}(C_{j,i}) = \phi$

Hence the result.

Lemma: 2.3

Let G be a Graph with $|V(G)| = n$, then

(i) $d_{ve}(G, i) = 0$ iff $i < \gamma_{ve}(G)$ or $i > n$

(ii) If G is connected, then $d_{ve}(G, n) = 1$ and $d_{ve}(G, n-1) = n$

(iii) $D_{ve}(G, x)$ has no constant term.

(iv) $D_{ve}(G, x)$ is a strictly increasing function in $[0, \infty)$.

Proof is obvious.

Lemma: 2.4

(i) If $D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-4}, {}_{i-1}) = \phi$, then $D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{i-1}) = \phi$.

(ii) If $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$, then $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$.

(iii) If $D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{i-1}) = D_{ve}(C_{n-4}, {}_{i-1}) = \phi$ then $D_{ve}(C_n, {}_i) = \phi$.

Proof

(i) Since, $D_{ve}(C_{n-1}, {}_{i-1}) = \phi$, by lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$.

Also, since $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$, by lemma 2.2, $i-1 > n-4$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$.

Therefore,

$$i-1 > n-2 \text{ or } i-1 < \left\lceil \frac{n-2}{4} \right\rceil \text{ and } i-1 > n-3 \text{ or } i-1 < \left\lceil \frac{n-3}{4} \right\rceil.$$

$\therefore D_{ve}(C_{n-2}, {}_{i-1}) = \phi$ and $D_{ve}(C_{n-3}, {}_{i-1}) = \phi$

(ii) $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$

$$\therefore \left\lceil \frac{n-1}{4} \right\rceil \leq i-1 \leq n-1 \text{ and}$$

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 \leq n-4$$

Therefore,

$$\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 \leq n-4$$

$$\therefore \left\lceil \frac{n-3}{4} \right\rceil \leq i-1 \leq n-3 \text{ and}$$

$$\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-2$$

$\therefore$ $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$

(iv) We have, $D_{ve}(C_{n-1}, {}_{i-1}) = \phi$

$$\therefore i-1 > n-1 \text{ or } i-1 < \left\lceil \frac{n-1}{4} \right\rceil D_{ve}(C_{n-2}, {}_{i-1}) = \phi$$

$$\therefore i-1 > n-2 \text{ or } i-1 < \left\lceil \frac{n-2}{4} \right\rceil$$

$$\text{Therefore, } i-1 > n-1 \text{ or } i-1 < \left\lceil \frac{n-4}{4} \right\rceil$$

$$\therefore i > n \text{ or } i < \left\lceil \frac{n-4}{4} \right\rceil + 1 \leq \left\lceil \frac{n}{4} \right\rceil$$

$$\therefore i > n \text{ or } i \leq \left\lceil \frac{n}{4} \right\rceil$$

$\therefore D_{ve}(C_n, {}_i) = \phi$.

Hence the result.

Lemma: 2.5

Let C_n, $n \geq 3$ be the cycle with $V(C_n) = n$ suppose that $D_{ve}(C_n, i) \neq \phi$, then we have

(i) $D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-2}, {}_{i-1}) = \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) = \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$ iff $n = 4k$ and $i = k$, for some $k \in N$.

(ii) $D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{i-1}) =$ $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$ and $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$ iff $i = n$.

(iii) $D_{ve}(C_{n-1}, {}_{i-1}) = \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$ iff $n = 4k+2$ and $i = \left\lceil \frac{4k+2}{4} \right\rceil$ for some $k \in N$.

(iv) $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$, and $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$ iff $i = n-2$.

(v) $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$, iff, $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i \leq n-3$.

(vi) $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) = \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$, iff, $i = n-1$.

Proof:

(i) Since, $D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{i-1}) = \phi$, by lemma 2.2, $i-1 > n-1$ or

$$i-1 < \left\lceil \frac{n-3}{4} \right\rceil$$

Also, since, $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$,

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 \leq n-4$$

Hence $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < \left\lceil \frac{n-3}{4} \right\rceil$

$$\therefore \left\lceil \frac{n-4}{4} \right\rceil + 1 \leq i < \left\lceil \frac{n-3}{4} \right\rceil + 1 \text{ ----------(1)}$$

Suppose, $n = 4k$, $\left\lceil \frac{n-4}{4} \right\rceil + 1 = k - 1 + 1 = k$

$$\left\lceil \frac{n-3}{4} \right\rceil + 1 = k + 1$$

Hence, $k \leq i < k + 1$

$\therefore i = k$

if $n \neq 4k$, then from (1),

We have an inequality of the form

$s \leq i < s$

$\Rightarrow \Leftarrow$

Hence $n = 4k$

Conversly, Assume that, $n = 4k$ and $i = k$, $k \in N$

$$\therefore \quad i = k = \frac{n}{4}$$

$$\therefore \quad i = \frac{n}{4}$$

$$\therefore \quad i-1 = \frac{n}{4} - 1$$

$$i-1 = \frac{n-4}{4} < \frac{n-1}{4}$$

$$\therefore \quad i-1 < \left\lceil \frac{n-1}{4} \right\rceil$$

$\therefore D_{ve}(C_{n-1}, {}_{i-1}) = \phi$,

$$i-1 = \frac{n-4}{4} < \frac{n-2}{4}$$

$$\therefore \quad i-1 < \left\lceil \frac{n-2}{4} \right\rceil$$

$\therefore D_{ve}(C_{n-2}, {}_{i-1}) = \phi$, and

$$i-1 = \frac{n-4}{4} < \frac{n-3}{4}$$

$$\therefore \quad i-1 < \left\lceil \frac{n-3}{4} \right\rceil$$

$\therefore D_{ve}(C_{n-3}, {}_{i-1}) = \phi$,

$$i = \frac{n}{4}$$

$$i - 1 = \frac{n}{4} - 1 = \frac{n-4}{4} < n - 4$$

$\therefore$ $i - 1 < n - 4$

$\therefore$ $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$.

(ii) Since $D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{i-1}) = D_{ve}(C_{n-4}, {}_{i-1}) = \phi$ $\therefore$ by lemma, 2.2

$$i - 1 > n - 2 \quad \text{or} \quad i - 1 < \left\lceil \frac{n-4}{4} \right\rceil$$

if $i - 1 < \left\lceil \frac{n-4}{4} \right\rceil$, then $i - 1 < \left\lceil \frac{n-4}{4} \right\rceil < \left\lceil \frac{n-1}{4} \right\rceil$ $\therefore$ $i - 1 < \left\lceil \frac{n-1}{4} \right\rceil$

$\therefore$ $D_{ve}(C_{n-1}, {}_{i-1}) = \phi$,

$\Rightarrow \Leftarrow$

$\therefore i - 1 > n - 2$

$\therefore i > n - 1$

$\therefore i = n, n + 1, \ldots$

Since, $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$

$\therefore i - 1 \leq n - 1$

$\therefore i \leq n$

$\therefore$ $i > n - 1$ and $i \leq n$.

$\therefore$ $i = n$

Conversly, If $i = n$, then by lemma 2.2.

We have,

$D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-2}, {}_{n-1}) = \phi$, $D_{ve}(C_{n-3}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{n-1}) = \phi$,

$D_{ve}(C_{n-4}, {}_{i-1}) = D_{ve}(C_{n-4}, {}_{n-1}) = \phi$, $D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-1}, {}_{n-1}) \neq \phi$

(iii) Since, $D_{ve}(C_{n-1}, {}_{i-1}) = \phi$, by lemma 2.2

$i - 1 > n - 1$ or $i - 1 < \left\lceil \frac{n-1}{4} \right\rceil$ if $i - 1 > n - 1$, then $i - 1 > n - 2$

and by lemma 2.2, $D_{ve}(C_{n-2}, {}_{i-1}) = \phi$,

$D_{ve}(C_{n-3}, {}_{i-1}) = \phi$, $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$,

$\Rightarrow \Leftarrow$

So, $i - 1 < \left\lceil \frac{n-1}{4} \right\rceil$

$\therefore$ $i < \left\lceil \frac{n-1}{4} \right\rceil + 1$

Since, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$

$\therefore$ $i - 1 \geq \left\lceil \frac{n-2}{4} \right\rceil$

$\therefore$ $\left\lceil \frac{n-2}{4} \right\rceil \leq i - 1$

$\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i$

and $\quad i \; < \; \left\lceil \frac{n-1}{4} \right\rceil + 1$

$$\therefore \left\lceil \frac{n-2}{4} \right\rceil + 1 \;\leq\; i \;\square\!< \left\lceil \frac{n-1}{4} \right\rceil + 1 \qquad \text{- - - - - (1)}$$

Suppose n = 4k + 2

$$\left\lceil \frac{n-2}{4} \right\rceil + 1 \;=\; \left\lceil \frac{4k+2-2}{4} \right\rceil + 1 = \; k \; + 1$$

$$\left\lceil \frac{n-1}{4} \right\rceil + 1 \;=\; \left\lceil \frac{4k+2-1}{4} \right\rceil + 1 = \left\lceil \frac{4k+1}{4} \right\rceil + 1$$

$$= k + 1 + 1$$

$$= k + 2$$

$$\therefore\; k + 1 \leq \; i < \; k + 2$$

$$\therefore\; i = k + 1 = \left\lceil \frac{4k+2}{4} \right\rceil$$

if $n \neq 4k + 2$, then from (1), we have an inequality of the form $s \leq i < s$, $\Rightarrow\Leftarrow$

Hence, n = 4k + 2 and $i = \; k + 1 = \left\lceil \frac{4k+2}{4} \right\rceil$

Conversly, Assume that n = 4k + 2 and

$$i = k + 1 = \left\lceil \frac{4k+2}{4} \right\rceil, k \in N$$

$$i = k + 1$$

$$\therefore\; i - 1 = k$$

$$= \frac{n-2}{4}$$

$$i - 1 = \frac{n-2}{4} < \frac{n-1}{4}$$

$$\therefore\; i - 1 \; < \left\lceil \frac{n-1}{4} \right\rceil$$

Therefore, $D_{ve}(C_{n-1\,,\,i-1}) \;=\; \phi$□

$$i - 1 = \frac{n-2}{4}$$

$$\therefore\; i - 1 \;= \left\lceil \frac{n-2}{4} \right\rceil$$

Therefore , $D_{ve}(C_{n-2\,,\,i-1}) \;\neq\; \phi$□

$$i - 1 = \frac{n-2}{4} > \frac{n-3}{4}$$

$$\therefore\; i - 1 \; > \left\lceil \frac{n-3}{4} \right\rceil$$

Therefore, $D_{ve}(C_{n-3\,,\,i-1}) \;\neq\; \phi$□

$$i - 1 = \frac{n-2}{4} > \frac{n-4}{4} \quad \therefore \; i - 1 \geq \left\lceil \frac{n-4}{4} \right\rceil$$

Therefore, $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$.

(iv) Since, $D_{ve}(C_{n-4}, {}_{i-1}) = \phi$, by lemma 2.2.

$$i - 1 > n - 4 \text{ or } i - 1 < \left\lceil \frac{n-4}{4} \right\rceil$$

$D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$, by lemma 2.2.

$$\left\lceil \frac{n-3}{4} \right\rceil \leq i - 1 \leq n - 3$$

$$\therefore \left\lceil \frac{n-3}{4} \right\rceil + 1 \leq i \leq n - 2.$$

Therefore, $i - 1 < \left\lceil \frac{n-4}{4} \right\rceil$ is not possible

Hence, $i - 1 > n - 4$

$\therefore$ $i > n - 3$

$\therefore$ $1 \geq n - 2$ but $i \leq n - 2$

Therefore, $i = n - 2$

conversely, if $i = n - 2$, then

$D_{ve}(C_{n-1}, {}_{i-1}) = D_{ve}(C_{n-1}, {}_{n-3}) \neq \phi$

$D_{ve}(C_{n-2}, {}_{i-1}) = D_{ve}(C_{n-2}, {}_{n-3}) \neq \phi$

$D_{ve}(C_{n-3}, {}_{i-1}) = D_{ve}(C_{n-3}, {}_{n-3}) \neq \phi$

and, $D_{ve}(C_{n-4}, {}_{i-1}) = D_{ve}(C_{n-4}, {}_{n-3}) = \phi$

(v) Since, $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$,

$D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$ and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$.

we have,
$$\left\lceil \frac{n-1}{4} \right\rceil \leq i - 1 \leq n - 1$$

$$\left\lceil \frac{n-2}{4} \right\rceil \leq i - 1 \leq n - 2$$

$$\left\lceil \frac{n-3}{4} \right\rceil \leq i - 1 \leq n - 3$$

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i - 1 \leq n - 4$$

So, $\left\lceil \frac{n-1}{4} \right\rceil \leq i - 1 \leq n - 4$

Hence, $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i \leq n - 3$

Conversly, if $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i \leq n - 3$ then by lemma 2.2,

$D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$,

$D_{ve}(C_{n-3}, {}_{i-1}) \neq \phi$, and $D_{ve}(C_{n-4}, {}_{i-1}) \neq \phi$.

(vi) since, $D_{ve}(C_{n-1}, {}_{i-1}) \neq \phi$, $D_{ve}(C_{n-2}, {}_{i-1}) \neq \phi$, by lemma 2.2

$\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$ -----------(1)

$\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$

Since, $D_{ve}(C_{n-3,\,i-1}) = \phi$, then by lemma 2.2

$\therefore$ $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$ ------- (2)

$D_{ve}(C_{n-4,\,i-1}) = \phi$, then by lemma 2.2

$\therefore$ $i-1 > n-4$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ ------- (3)

From (1), if $i-1 = \left\lceil \frac{n-1}{4} \right\rceil$

$D_{ve}(C_{n-3,i-1}) = D_{ve}(C_{n-3}, \left\lceil \frac{n-1}{4} \right\rceil) \neq \phi$,

$\therefore$ contradiction.

$\therefore i-1 \neq \left\lceil \frac{n-1}{4} \right\rceil$

$\therefore$ $i-1 \le n-1$

$i \le n$

$\therefore$ $i = n, n-1, n-2, \ldots$

if $i = n$,

$D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-1,\,n-1}) \neq \phi$,

$D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-2,\,n-1}) = \phi$, a contradiction. $\therefore$ $i \neq n$,

if $i = n-1$,

$D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-1,\,n-2}) \neq \phi$,

$D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-2,\,n-2}) \neq \phi$

$D_{ve}(C_{n-3,\,i-1}) = D_{ve}(C_{n-3,\,n-2}) = \phi$,

$D_{ve}(C_{n-4,\,i-1}) = D_{ve}(C_{n-4,\,n-2}) = \phi$

if $i = n-2$,

$D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-1,\,n-3}) \neq \phi$,

$D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-2,\,n-3}) \neq \phi$

$D_{ve}(C_{n-3,i-1}) = D_{ve}(C_{n-3,\,n-3}) \neq \phi$,

a contradiction

$\therefore$ $i \neq n-2$ $\therefore$ $i = n-1$

conversly, if $i = n-1$

$\therefore$ $D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-1,\,n-2}) \neq \phi$,

$D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-2,\,n-2}) \neq \phi$,

$D_{ve}(C_{n-3,\,i-1}) = D_{ve}(C_{n-3,\,n-2}) = \phi$,

$D_{ve}(C_{n-4,\,i-1}) = D_{ve}(C_{n-4,\,n-2}) = \phi$

Theorem 2.6

For every $n \ge 5$ and $i \ge \left\lceil \frac{n}{4} \right\rceil$

(i) If $D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-2,\,i-1})$ $= D_{ve}(C_{n-3,\,i-1}) = \phi$ and $D_{ve}(C_{n-4,\,i-1}) \neq \phi$, then, $D_{ve}(C_{n,\,i}) = \{\{1, 5, 9, \ldots, n-3\}, \{2, 6, 10, \ldots, n-2\}, \{3, 7, 11, \ldots, n-1\}, \{4, 8, 12, \ldots, n\}\}$

(ii) If $D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-3,\,i-1}) = D_{ve}(C_{n-4,\,i-1}) = \phi$, and $D_{ve}(C_{n-1,\,i-1}) \neq \phi$, then $D_{ve}(C_{n,\,i}) = \{[n]\}$

(iii) $D_{ve}(C_{n-4,\,i-1}) = \phi$, $D_{ve}(C_{n-3,\,i-1}) \neq \phi$, $D_{ve}(C_{n-2,\,i-1}) \neq \phi$ and $D_{ve}(C_{n-1,\,i-1}) \neq \phi$, then $D_{ve}(C_{n,i})$ $= \{[n] - \{x, y\} \mid x, y \in [n]\}$.

(iv) If $D_{ve}(C_{n-1,\,i-1}) = \phi$, $D_{ve}(C_{n-2,\,i-1}) \neq \phi$, $D_{ve}(C_{n-3,\,i-1}) \neq \phi$ and $D_{ve}(C_{n-4,\,i-1}) \neq \phi$, then

$D_{ve}(C_{n,i}) = \{\{1, 5, 9, \ldots, n-3\},$
$\{2, 6, 10, \ldots, n-2\},$
$\{3, 7, 11, \ldots, n-1\},$
$\{4, 8, 12, \ldots, n\}\} \cup$

$\{X_2 \cup \{n\} \mid X_2 \in D_{ve}(C_{n-2,\,i-1})\} \cup \{X_3 \cup \{n-1\} \mid X_3 \in D_{ve}(C_{n-3,\,i-1})\} \cup \{X_4 \cup \left\{\begin{array}{l} n-2 \text{ if } 2 \in X_4 \\ n-3 \text{ if } 2 \notin X_4 \end{array}\right. \mid X_4 \in D_{ve}(C_{n-4,\,i-1})\}$

(v) If $D_{ve}(C_{n-1,\,i-1}) \neq \phi$, $D_{ve}(C_{n-2,\,i-1}) \neq \phi$, $D_{ve}(C_{n-3,\,i-1}) \neq \phi$ and $D_{ve}(C_{n-4,\,i-1}) \neq \phi$, then $D_{ve}(C_{n,i}) = \{X_1 \cup \{n\} \mid X_1 \in D_{ve}(C_{n-1,i-1})\}$
$\cup \{X_2 \cup \{n-1\} \mid X_2 \in D_{ve}(C_{n-2,\,i-1})\}$
$\cup \{X_3 \cup \{n-2\} \mid X_3 \in D_{ve}(C_{n-3,\,i-1})\}$
$\cup \{X_4 \cup \{n-3\} \mid X_4 \in D_{ve}(C_{n-4,\,i-1})\}$

Proof:

(i) If $D_{ve}(C_{n-1,\,i-1}) = D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-3,\,i-1}) = \phi$, and $D_{ve}(C_{n-4,\,i-1}) \neq \phi$, then by lemma 2.5 (i), $n = 4k$ and $i = k \in N$

$\therefore k = \frac{n}{4}$

$\therefore i = k = \frac{n}{4}$

$\therefore D_{ve}(C_{n,i}) = D_{ve}(C_n, \frac{n}{4})$

Clearly, $\{\{1, 5, 9, \ldots, n-3\}, \{2, 6, 10, \ldots, n-2\}, \{3, 7, 11, \ldots, n-1\}, \{4, 8, 12, \ldots, n\}\}$ is a vertex-edge dominating set with $i = \frac{n}{4}$ elements

Also, no other set of cardinality $\frac{n}{4}$ is a vertex-edge dominating set.

Therefore, $D_{ve}(C_{n,i}) = D_{ve}(C_n, \frac{n}{4})$

$= \{\{1, 5, 9, \ldots, n-3\}, \{2, 6, 10, \ldots, n-2\},$
$\{3, 7, 11, \ldots, n-1\}, \{4, 8, 12, \ldots, n\}\}$

(ii) We have, $D_{ve}(C_{n-2,\,i-1}) = D_{ve}(C_{n-3,\,i-1})$ $= D_{ve}(C_{n-4,\,i-1}) = \phi$, and $D_{ve}(C_{n-1,\,i-1}) \neq \phi$, by lemma 2.5 (ii) we have $i = n$.

So, $D_{ve}(C_{n,i}) = D_{ve}(C_n, {}_n)$
$= \{\{1, 2, 3, \ldots, n\}\} = \{[n]\}$

(iii) We have $D_{ve}(C_{n-4,\,i-1}) = \phi$,
$D_{ve}(C_{n-3,\,i-1}) \neq \phi$,
$D_{ve}(C_{n-2,\,i-1}) \neq \phi$
$D_{ve}(C_{n-1,\,i-1}) \neq \phi$,

by lemma 2.5 (iv), $i = n - 2$

$D_{ve}(C_{n,i}) = D_{ve}(C_n, {}_{n-2})$

If we have n vertices, we remove two vertices that will cover all the vertices and edges.

$D_{ve}(C_{n,i}) = D_{ve}(C_n, {}_{n-2}) = \{[n] - \{x, y\} \mid x, y \in [n]\}$

(iv) We have $D_{ve}(C_{n-1}, i-1) = \phi$,
$D_{ve}(C_{n-2}, i-1) \neq \phi$
$D_{ve}(C_{n-3}, i-1) \neq \phi$ and $D_{ve}(C_{n-4}, i-1) \neq \phi$, and
By lemma 2.5 (iii),

$n = 4k + 2$ and $i = \left\lceil \frac{4k+2}{4} \right\rceil = k + 1$, $k \in N$ we denote the families

$Y_1 = \{\{1, 5, 9, . . . , 4k - 3, 4k + 1\}, \{2, 6, 10, . . . , 4k - 2, 4k + 2\}, \{3, 7, 11, . . . , 4k - 1, 4k + 3\}, \{4, 8, 12, . . . , 4k, 4k + 4\}\}$ and

$Y_2 = \{ X_2 \cup \{4k + 2\} \mid X_2 \in D_{ve}(C_{n-2, i-1})\} \cup \{X_3 \cup \{4k + 1\} \mid X_3 \in D_{ve}(C_{n-3, i-1})\}$

$\cup \left\{ X_4 \cup \left\{ \begin{array}{l} 4k \text{ if } 2 \in X_4 \\ 4k-1 \text{ if } 2 \notin X_4 \end{array} \right\} \middle| X_4 \in D_{ve}(C_{n-4, i-1}) \right\}$

We have to prove that $D_{ve}(C_{4k+2, k+1}) = Y_1 \cup Y_2$

Since $D_{ve}(C_{4k, k}) = \{\{1, 5, 9, . . . , 4k - 3, 4k + 1\},$
$\{2, 6, 10, . . . , 4k - 2, 4k + 2\},$
$\{3, 7, 11, . . . , 4k - 1, 4k + 3\},$
$\{4, 8, 12, . . . , 4k, 4k + 4\}\}$

then $Y_1 \subseteq D_{ve}(C_{4k+2, k+1})$

Also, $Y_2 \subseteq D_{ve}(C_{4k+2, k+1})$

Therefore, $Y_1 \cup Y_2 \subseteq D_{ve}(C_{4k+2, k+1})$

Now, Let $Y \in D_{ve}(C_{4k+2, k+1})$

Suppose $Y \in Y_1$. Then $Y \in Y_1 \cup Y_2$

Now, assume that $Y \notin Y_1$.

By lemma 2.3 [7] atleast one of the vertices labelled $4k + 2, 4k + 1, 4k$ or $4k - 1$ is in Y. If $4k + 2 \in Y$, then

By Lemma 2.3 [7], atleast one vertex labelled 1, 2 or 3 and $4k + 1$, $4k$ or $4k - 1$ is in Y. If $4k + 1$ or $4k$ is in Y. then $Y - \{4k + 2\} \in D_{ve}(C_{4k+1, k})$, a contradiction. because $D_{ve}(C_{4k+1, k}) = \phi$ hence $4k - 1 \in Y$, $4k \notin Y$ and $4k + 1 \notin Y$.

Therefore, $Y = X \cup \{4k + 2\}$ for some $X \in D_{ve}(C_{4k, k})$

Hence $Y \in Y_1$, similar arguments follow if 1 or 2 or $3 \in Y$.

Now, suppose that $4k + 1 \in Y$ and $4k + 2 \notin Y$

By lemma 2.3 [7], atleast one vertex labelled $4k$, $4k - 1$, $4k - 2$, $4k - 3$ is in Y. or $1, 2, 3 \in Y$. If $4k \in Y$, then $Y - \{4k + 1\} \in D_{ve}(C_{4k, k}) = \{\{1, 5, 9, ..., 4k - 3, 4k + 1\},\{2, 6, 10, . . . , 4k - 2 , 4k + 2\}, \{3, 7, 11, . . . , 4k - 1 , 4k + 3\}, \{4, 8, 12, . . . , 4k, 4k + 4\}\}$ a contradiction. Because $4k \notin X$ for all $X \in D_{ve}(C_{4k, k})$.

Therefore, $4k - 1$ or $4k - 2$ is in Y. but $4k \notin Y$. Thus $Y = X \cup \{4k + 1\}$ for some $X \in D_{ve}(C_{4k+1, k})$. Similar argument follows if 1 or 2 or $3 \in Y$.

So, $D_{ve}(C_{4k+2, k+1}) \subseteq Y_1 \cup Y_2$

Hence $D_{ve}(C_{4k+2, k+1}) = Y_1 \cup Y_2$

$\therefore D_{ve}(C_{n, i}) = \{\{1, 5, 9, . . . , n - 3\},$
$\{2, 6, 10, . . . , n - 2\},$
$\{3, 7, 11, . . . , n - 1\},$
$\{4, 8, 12, ..., n\}\} \cup$
$\{ X_2 \cup \{n\}\ X_2 \in D_{ve}(C_{n-2, i-1}) \} \cup$
$\{X_3 \cup \{n - 1\} \mid X_3 \in D_{ve}(C_{n-3, i-1})\} \cup$
$\left\{ X_4 \cup \left\{ \begin{array}{l} n-2 \text{ if } 2 \in X_4 \\ n-3 \text{ if } 2 \notin X_4 \end{array} \right. \middle| X_4 \in D_{ve}(C_{n-4, i-1}) \right\}$

(v) $D_{ve}(C_{n-1, i-1}) \neq \phi$, $D_{ve}(C_{n-2, i-1}) \neq \phi$, $D_{ve}(C_{n-3, i-1}) \neq \phi$, and $D_{ve}(C_{n-4, i-1}) \neq \phi$

Let $X_1 \in D_{ve}(C_{n-1, i-1})$, so atleast one vertex labeled $n - 1$ or $n - 2$ or $n - 3$ or $n - 4$ is in X_1.

If $n - 1$ or $n - 2$ or $n - 3$ or $n - 4 \in X_1$, then $X_1 \cup \{n\} \in D_{ve}(C_{n, i})$.

Let $X_2 \in D_{ve}(C_{n-2, i-1})$, then $n - 2$, $n - 3$ or $n - 4$ or $n - 5$ is in X_2.

If $n - 2$ or $n - 3$ or $n - 4$ or $n - 5 \in X_2$, then

$X_2 \cup \{ n - 1 \} \in D_{ve}(C_{n, i})$.

Now, Let $X_3 \in D_{ve}(C_{n-3, i-1})$, then $n - 3$, $n - 4$, $n - 5$ or $n - 6$ is in X_3.

If $n - 3$ or $n - 4$ or $n - 5$ or $n - 6 \in X_3$ then

$X_3 \cup \{ n - 2 \} \in D_{ve}(C_{n, i})$.

Now, Let $X_4 \in D_{ve}(C_{n-4, i-1})$, then $n - 4$ or $n - 5$ or $n - 6$ or $n - 7$ is in X_4.

If $n - 4$, $n - 5$, $n - 6$ or $n - 7 \in X_4$, then

$X_4 \cup \{ n - 3 \} \in D_{ve}(C_{n, i})$.

Thus we have,

$\{X_1 \cup \{n\} | X_1 \in D_{ve}(C_{n-1}, i-1),$

$\cup \{X_2 \cup \{n-1\} | X_2 \in D_{ve}(C_{n-2}, i-1),$

$\cup X_3 \cup \{n-2\} | X_3 \in D_{ve}(C_{n-3}, i-1)$

$\cup X_4 \cup \{n-4\} | X_4 \in D_{ve}(C_{n-4}, i-1)\} \subseteq D_{ve}(C_{n,i})$

Let $Y \in D_{ve}(C_{n,i})$, then n, n – 1 , n – 2 , or n – 3 $\in Y$

Now, suppose that $n \in Y$, $n-1 \in Y$ then by lemma 2.3[7], atleast one vertex labeled n – 2, n – 3 or n – 4 is in Y.

If $n-2 \in Y$, then $Y = X_3 \cup \{n-2\}$ for some

$X_3 \in D_{ve}(C_{n-3}, i-1)$

Now, suppose that $n-5 \in Y$, $n-4 \in Y$, $n-3 \in Y$, then by lemma 2.3 [7], atleast one vertex labelled n – 6, n – 7 or n – 8 is in Y.

If $n-4 \in Y$, then $Y = X_2 \cup \{n-1\}$ for some

$X_2 \in D_{ve}(C_{n-2}, i-1)$

Now, suppose that $n-8, n-9, n-10 \in Y$,

then By lemma 2.3 [7] atleast one vertex labelled n – 11, n – 12 or n – 13 is in Y.

If $n-8 \in Y$, then $Y = X_1 \cup \{n\}$ for some

$X_1 \in D_{ve}(C_{n-1}, i-1)$.

So, $D_{ve}(C_n, i) \subseteq \{X_1 \cup \{n\} | X_1 \in D_{ve}(C_{n-1}, i-1),$

$\cup \{X_2 \cup \{n-1\} | X_2 \in D_{ve}(C_{n-2}, i-1),$

$\cup X_3 \cup \{n-2\} | X_3 \in D_{ve}(C_{n-3}, i-1)$

$\cup X_4 \cup \{n-4\} | X_4 \in D_{ve}(C_{n-4}, i-1)\}$

$\therefore D_{ve}(C_n, i) = \{X_1 \cup \{n\} | X_1 \in D_{ve}(C_{n-1}, i-1),$

$\cup \{X_2 \cup \{n-1\} | X_2 \in D_{ve}(C_{n-2}, i-1),$

$\cup X_3 \cup \{n-2\} | X_3 \in D_{ve}(C_{n-3}, i-1)$

$\cup X_4 \cup \{n-4\} | X_4 \in D_{ve}(C_{n-4}, i-1)\}$

$d_{ve}(C_{n,j})$, The Number of Vertex- Edge dominating sets of C_n with cardinality j

n \ j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	1														
2	2	1													
3	3	3	1												
4	4	6	4	1											
5	0	10	10	5	1										
6	0	9	20	15	6	1									
7	0	7	28	35	21	7	1								
8	0	4	32	62	56	28	8	1							
9	0	0	30	90	117	84	36	9	1						
10	0	0	20	110	202	200	120	45	10	1					
11	0	0	11	110	297	396	319	165	55	11	1				
12	0	0	4	93	372	672	708	483	220	66	12	1			
13	0	0	0	65	403	988	1352	1183	702	286	78	13	1		
14	0	0	0	35	378	1274	2256	2499	1876	987	364	91	14	1	
15	0	0	0	15	303	1450	3330	4635	4330	2853	1350	455	105	15	1

3. Vertex-Edge Domination Polynomial of Cycle

Theorem:3.1

1. If $D_{ve}(C_n,i)$ is the family of vertex-edge dominating sets with cardinality i of C_n, then

$$|D_{ve}(C_n, i)| = |D_{ve}(C_{n-1}, i-1)| + |D_{ve}(C_{n-2}, i-1)| + |D_{ve}(C_{n-3}, i-1)| + |D_{ve}(C_{n-4}, i-1)|$$

2. For every $n \geq 5$,

$$D_{ve}(C_n, x) = x\,[\,D_{ve}(C_{n-1}, x) + D_{ve}(C_{n-2}, x) + D_{ve}(C_{n-3}, x) + D_{ve}(C_{n-4}, x)\,]$$

with initial values ,

$D_{ve}(C_1, x) = x$

$D_{ve}(C_2, x) = x^2 + 2x$

$D_{ve}(C_3, x) = x^3 + 3x^2 + 3x$

$D_{ve}(C_4, x) = x^4 + 4x^3 + 6x^2 + 4x$

Proof:

1. From Theorem 2.6, we consider the cases given below, where $i \geq \left\lceil \frac{n}{4} \right\rceil$ and $n \geq 5$

(i) If $D_{ve}(C_{n-1}, i-1) = D_{ve}(C_{n-2}, i-1) = D_{ve}(C_{n-3}, i-1) = \phi$ and $D_{ve}(C_{n-4}, i-1) \neq \phi$, then

$D_{ve}(C_n, i) = \{\{1, 5, 9, \ldots, n-3\}, \{2, 6, 10, \ldots, n-2\}, \{3, 7, 11, \ldots, n-1\}, \{4, 8, 12, \ldots, n\}\}$.

(ii) If $D_{ve}(C_{n-2}, i-1) = D_{ve}(C_{n-3}, i-1) = D_{ve}(C_{n-4}, i-1) = \phi$ and $D_{ve}(C_{n-1}, i-1) \neq \phi$, then $D_{ve}(C_n, i) = \{[n]\}$

(iii) If $D_{ve}(C_{n-4}, i-1) = \phi$, $D_{ve}(C_{n-3}, i-1) \neq \phi$, $= D_{ve}(C_{n-2}, i-1) \neq \phi$ and $D_{ve}(C_{n-1}, i-1) \neq \phi$, then $D_{ve}(C_n, i) = \{ [x] - \{x, y\} \mid x, y \in [n]\}$

(iv) If $D_{ve}(C_{n-1}, i-1) = \phi$, $D_{ve}(C_{n-2}, i-1) \neq \phi$, $= D_{ve}(C_{n-3}, i-1) \neq \phi$ and $D_{ve}(C_{n-4}, i-1) \neq \phi$, then

$D_{ve}(C_n, i) = \{\{1, 5, 9, \ldots, n-3\}, \{2, 6, 10, \ldots, n-2\}, \{3, 7, 11, \ldots, n-1\}, \{4, 8, 12, \ldots, n]$

$\cup \{X_2 \cup \{n\} \mid X_2 \in D_{ve}(C_{n-2}, i-1)\}$

$\cup \{X_3 \cup \{n-1\} \mid X_3 \in D_{ve}(C_{n-3}, i-1\}$

$\cup \left\{X_4 \cup \begin{cases} n-2 & \text{if } 2 \in X_4 \\ n-3 & \text{if } 2 \notin X_4 \end{cases} \;\middle|\; X_4 \in D_{ve}(C_{n-4}, i-1)\right\}$

(v) $D_{ve}(C_{n-1}, i-1) \neq \phi$, $D_{ve}(C_{n-2}, i-1) \neq \phi$, $D_{ve}(C_{n-3}, i-1) \neq \phi$ and $D_{ve}(C_{n-4}, i-1) \neq \phi$, then $D_{ve}(C_{n,i}) = |\{X_1 \cup \{n\} \mid X_1 \in D_{ve}(C_{n-1}, i-1\}\}$

$\cup \{X_2 \cup \{n-1\} \mid X_2 \in D_{ve}(C_{n-2}, i-1\}$

$\cup \{X_3 \cup \{n-2\} \mid X_3 \in D_{ve}(C_{n-3}, i-1\}$

$\cup \{X_4 \cup \{n-3\} \mid X_4 \in D_{ve}(C_{n-4}, i-1\}\}$.

From the above constitution, in each case, we obtain that,

$$|D_{ve}(C_n, i)| = |D_{ve}(C_{n-1}, i-1)| + |D_{ve}(C_{n-2}, i-1)| + |D_{ve}(C_{n-3}, i-1)| + |D_{ve}(C_{n-4}, i-1)|$$

(2). By definition,

$$D_{ve}(C_n, x) = \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} d_{ve}(C_{n,i})\, x^i$$

$$= x \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n,i}) x^{i-1} \text{ by using part (1)}$$

$$= x\left(\sum_{i=\lceil \frac{n}{4} \rceil}^{n} \begin{pmatrix} (d_{ve}(C_{n-1}, i-1) + d_{ve}(C_{n-2}, i-1) \\ + d_{ve}(C_{n-3}, i-1) + d_{ve}(C_{n-4}, i-1) \end{pmatrix} x^{i-1} \right) = x\left(\sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n-1,i-1}) x^{i-1} \right.$$

$$+ \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n-2,i-1}) x^{i-1}$$

$$+ \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n-3,i-1}) x^{i-1}$$

$$\left. + \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n-4}, i-1) x^{i-1} \right)$$

$$= x [D_{ve}(C_{n-1}, x) + D_{ve}(C_{n-2}, x) + D_{ve}(C_{n-3}, x) + D_{ve}(C_{n-4}, x)]$$

The initial values are

$$D_{ve}(C_1, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{1} d_{ve}(C_{1,i}) x^i$$

$$= \sum_{i=0}^{1} d_{ve}(C_{1,i}) x^i = x$$

$$D_{ve}(C_2, x) = \sum_{i=\lceil \frac{2}{4} \rceil}^{2} d_{ve}(C_{2,i}) x^i$$

$$= \sum_{i=1}^{2} d_{ve}(C_{2,i}) x^i = x^2 + 2x$$

$$D_{ve}(C_3, x) = \sum_{i=\lceil \frac{3}{4} \rceil}^{3} d_{ve}(C_{3,i}) x^i$$

$$= \sum_{i=1}^{3} d_{ve}(C_{3,i}) x^i = x^3 + 3x^2 + 3x$$

$$D_{ve}(C_4, x) = \sum_{i=\lceil \frac{4}{4} \rceil}^{4} d_{ve}(C_{4,i}) x^i$$

$$= \sum_{i=1}^{4} d_{ve}(C_{4,i}) x^i = x^4 + 4x^3 + 6x^2 + 4x$$

Theorem:3.2

The following properties hold for the co-efficients of $D_{ve}(C_n, x)$:

(i) For every $n \in N$, $d_{ve}(C_{4n,n}) = 4$

(ii) For every $n \geq 5$, $j \geq \lceil \frac{n}{4} \rceil$

$d_{ve}(C_n, j) = d_{ve}(C_{n-1}, j-1)$
$+ d_{ve}(C_{n-2}, j-1)$
$+ d_{ve}(C_{n-3}, j-1)$
$+ d_{ve}(C_{n-4}, j-1)$

(iii) For every $n \in N$

$$d_{ve}(C_{4n+3}, n+1) = 4n+3$$

(iv) For every $n \geq 5$, $d_{ve}(C_{n,n}) = 1$

(v) For every $n \geq 5$, $d_{ve}(C_{n,n-1}) = n$

(vi) For every $n \geq 5$

$$d_{ve}(C_{n,n-2}) = \frac{(n-1)n}{2}$$

(vii) For every $n \geq 5$

$$d_{ve}(C_{n,n-3}) = \frac{n(n-1)(n-2)}{3!}$$

Proof:

(i) Since $D_{ve}(C_{4n}, n) = \{\{1, 5, 9, \ldots, 4n-3\}, \{2, 6, 10, \ldots, 4n-2\}, \{3, 7, 11, \ldots, 4n-1\}, \{4, 8, 12, \ldots, 4n\}\}$

$\therefore$ $d_{ve}(C_{4n,n}) = 4$

(ii) Proof follows from theorem 3.1

(iii) We prove this theorem by the method of induction on n.

Obviously, the result is true for n = 1.

Now, Suppose that the result is true for all natural Numbers less than n

$\therefore d_{ve}(C_{4n-1}, n) = 4n - 1.$

Now, we have to prove that the result is true for n.

$d_{ve}(C_{4n+3}, n+1) = d_{ve}(C_{4n+2}, n) + d_{ve}(C_{4n+1}, n) + d_{ve}(C_{4n}, n) + d_{ve}(C_{4n-1}, n)$ by (2)

$= 0 + 0 + 4 + 4n - 1$

$= 4n + 3$

$d_{ve}(C_{4n+3}, n+1) = 4n+3.$

by principle of mathematical induction, the result is true for all n, $n \in N$ (iv) Obviously, the result is true for n = 5. Now, suppose that the result is true for all natural Numbers less than n

$d_{ve}(C_{n-1, n-1}) = 1$. is true Now, We have to prove that the result is true for n.

$d_{ve}(C_{n,n}) = d_{ve}(C_{n-1, n-1}) + d_{ve}(C_{n-2, n-1})$
$+ d_{ve}(C_{n-3, n-1}) + d_{ve}(C_{n-4, n-1})$
$= 1 + 0 + 0 + 0 = 1$

Hence by principle of induction, the result is true for all n, $n \in N$ (v) We prove this result by the method of induction on n. Obviously, The result is true for n = 5

Assume that the result is true for all natural Numbers less than n

$d_{ve}(C_{n-1, n-2}) = n - 1$. is true

Now, We have to prove that the result is true for n.

$d_{ve}(C_{n, n-1}) = d_{ve}(C_{n-1, n-2}) + d_{ve}(C_{n-2, n-2})$
$+ d_{ve}(C_{n-3, n-2}) + d_{ve}(C_{n-4, n-2})$
$= n - 1 + 1 + 0 + 0 = n$

$d_{ve}(C_{n, n-1}) = n$

$\therefore$ by principle of induction, the result is true for all n, $n \in N$

(vi) We prove this theorem by method of induction on n. Obviously, the result is true for n = 5

Now, Assume that the result is true for all natural Numbers less than n.

$$d_{ve}(C_{n-1, n-3}) = \frac{(n-2)(n-1)}{2} \text{ is true}$$

Now, we have to prove that the result is true for n.

$d_{ve}(C_{n}, n-2) = d_{ve}(C_{n-1}, n-3)$
$+ d_{ve}(C_{n-2}, n-3)$
$+ d_{ve}(C_{n-3}, n-3)$

$$+ d_{ve}(C_{n-4}, n-3)$$

$$= \frac{(n-2)(n-1)}{2} + n - 2 + 1 + 0$$

$$= \frac{(n-1)n}{2}$$

by principle of induction, the result is true for all n, $n \in N$.

(vii) We prove this theorem by method of induction on n. Obviously, the result is true for n = 5 Assume that the result is true for all natural Numbers less than n.

$$d_{ve}(C_{n-1, n-4}) = \frac{(n-1)(n-2)(n-3)}{3!}$$

is true

Now, we have to prove that the result is true for n.

$$d_{ve}(C_{n}, n-3) = d_{ve}(C_{n-1}, n-4) + d_{ve}(C_{n-2}, n-4) + d_{ve}(C_{n-3}, n-4) + d_{ve}(C_{n-4}, n-4)$$

$$= \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-3)(n-2)}{2} + n - 3 + 1$$

$$= \frac{(n-1)(n-2)(n-3) + 3(n-3)(n-2) + 6(n-2)}{6}$$

$$= \frac{n(n-1)(n-2)}{3!}$$

Hence, by principle of mathematical induction, the result is true for all n, $n \geq 5$.

Theorem :3.3

(i) For every $j \geq 2$

$$\sum_{i=j}^{4j} d_{ve}(C_{i,j}) = 4 \sum_{i=j-1}^{4(j-1)} d_{ve}(C_{i,j-1})$$

(ii) If $S_n = \sum_{j=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n,j})$

then for every $n \geq 5$,

$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$ with initial values $S_1 = 1$, $S_2 = 3$, $S_3 = 7$, $S_4 = 15$.

(iii) For every $j \geq \lceil \frac{n}{4} \rceil$

$$d_{ve}(C_{n+1, j+1}) - d_{ve}(C_{n, j+1}) = d_{ve}(C_{n, j}) - d_{ve}(C_{n-4, j})$$

Proof:

(i) We prove this theorem by the method of induction on j.

Obviously, the result is true for j = 2

Suppose that the result is true for all j < k

Now,We have to prove that the result is true for j = k

$$\sum_{i=k}^{4k} d_{ve}(C_{i,k}) = \sum_{i=k}^{4k} [d_{ve}(C_{i-1,k-1}) + d_{ve}(C_{i-2,k-1}) + d_{ve}(C_{i-3,k-1}) + d_{ve}(C_{i-4,k-1})]$$

$$= \sum_{i=k}^{4k} d_{ve}(C_{i-1,k-1}) + \sum_{i=k}^{4k} d_{ve}(C_{i-2,k-1}) + \sum_{i=k}^{4k} d_{ve}(C_{i-3,k-1}) + \sum_{i=k}^{4k} d_{ve}(C_{i-4,k-1})$$

$$= 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i-1,\,k-2})$$

$$+ 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i-2,\,k-2})$$

$$+ 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i-3,\,k-2})$$

$$+ 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i-4,\,k-2})$$

$$= 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i,\,k-1})$$

$$\therefore \sum_{i=k}^{4k} d_{ve}(C_{i,k}) = 4\sum_{i=k-1}^{4k-4} d_{ve}(C_{i,\,k-1})$$

$\therefore$ The theorem is true for $j = k$.

Hence by principle of induction, the theorem is true for all j, $j \geq 2$.

(ii) We have, $S_n = \sum_{j=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n,j})$

$$= \sum_{j=\lceil \frac{n}{4} \rceil}^{n} [d_{ve}(C_{n-1,\,j-1}) + d_{ve}(C_{n-2,\,j-1})$$

$$+ d_{ve}(C_{n-3,\,j-1}) + d_{ve}(C_{n-4,\,j-1})]$$

$$= \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-1,\,j}) + \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-2,\,j})$$

$$+ \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-3,\,j}) + \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-4,\,j}) \quad \text{--(1)}$$

Consider $\sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-2,\,j})$

$$= \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-2} d_{ve}(C_{n-2,\,j}) + d_{ve}(C_{n-2,\,n-1})$$

$$= \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-2,\,j})$$

Consider $\sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-3,\,j})$

$$= \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-3} d_{ve}(C_{n-3,\,j}) + d_{ve}(C_{n-3,\,n-2})$$

$$+ d_{ve}(C_{n-3,\,n-1}) + \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-3} d_{ve}(C_{n-3,\,j}) + 0 + 0$$

$$= \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-3} d_{ve}(C_{n-3,\,j})$$

Similarly,

$$\sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} = \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-4} d_{ve}(C_{n-4,\,j})$$

Substitute in (1) we get

$$\sum_{j=\lceil \frac{n}{4} \rceil}^{n} d_{ve}(C_{n,\,j}) = \sum_{j=\lceil \frac{n}{4} \rceil-1}^{n-1} d_{ve}(C_{n-1,\,j})$$

$$+ \sum_{j=\lceil \frac{n}{4} \rceil -1}^{n-2} d_{ve}(C_{n-2,j}) + \sum_{j=\lceil \frac{n}{4} \rceil -1}^{n-3} d_{ve}(C_{n-3,j}$$

$$+ \sum_{j=\lceil \frac{n}{4} \rceil -1}^{n-4} d_{ve}(C_{n-4,j})$$

$$\therefore S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}.$$

iii) $d_{ve}(C_{n+1,j+1}) - d_{ve}(C_{n,j+1})$

$= [d_{ve}(C_{n,j}) + d_{ve}(C_{n-1,j})$
$+ d_{ve}(C_{n-2,j}) + d_{ve}(C_{n-3,j})]$
$- [(d_{ve}(C_{n-1,j}) + d_{ve}(C_{n-2,j})$
$+ d_{ve}(C_{n-3,j}) + d_{ve}(C_{n-4,j})]$.

$= [d_{ve}(C_{n,j}) + d_{ve}(C_{n-1,j}) + d_{ve}(C_{n-2,j})$
$+ d_{ve}(C_{n-3,j})] - d_{ve}(C_{n-1,j}) - d_{ve}(C_{n-2,j})$
$- d_{ve}(C_{n-3,j}) - d_{ve}(C_{n-4,j})$

$= d_{ve}(C_{n,j}) - d_{ve}(C_{n-4,j})$

Theorem: 3.4

For every $n \geq 5$, and $\lceil \frac{n}{4} \rceil \leq i \leq n$, $d_{ve}(C_{n,i})$ is the co-efficient of $u^n v^i$ in the Expansion of the function.

$$f(u,v) = \frac{u^5v^2\,[10 + 10v + 5v^2 + 9u + 10uv + 5uv^2 + u^3 + uv^3 + 7u^2 + 9u^2v + 5u^2v^2 + u^2v^3 + 4u^3 + 6u^3v + 4u^3v^2 + u^3v^3]}{1 - uv - u^2v - u^3v - u^4v}$$

Proof:

Set $f(u,v) = \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n,i})|\, u^n v^i$

by recursive formula for $d_{ve}(C_{n,i})$ in theorem 3.1 we can write f(u, v) in the following form.

$$f(u,v) = \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} [(|D_{ve}(C_{n-1,i-1})| + |D_{ve}(C_{n-2,i-1})| + |D_{ve}(C_{n-3,i-1})| + |D_{ve}(C_{n-4,i-1})|)]\, u^n v^i$$

$$= uv \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-1,i-1})|\, u^{n-1} v^{i-1}$$

$$+ u^2v \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-2,i-1})|\, u^{n-2} v^{i-1}$$

$$+ u^3v \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-3,i-1})|\, u^{n-3} v^{i-1}$$

$$+ u^4v \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-4,i-1})|\, u^{n-4} v^{i-1}$$

$$= uv(4u^4v + 6u^4v^2 + 4u^4v^3 + u^4v^4 + \sum_{n=6}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-1,i-1})|\, u^{n-1} v^{i-1})$$

$$+ u^2v(3u^3v + 3u^3v^2 + u^3v^3 + 4u^4v + 6u^4v^2 + 4u^4v^3 + u^4v^4 + \sum_{n=7}^{\infty} \sum_{i=2}^{\infty} |D_{ve}(C_{n-2,i-1})|\, u^{n-2} v^{i-1})$$

$$+ u^3v(2u^2v + u^2v^2 + 3u^3v + 3u^3v^2 + u^3v^3$$

$+ \ 4u^4v + \ 6u^4v^2 + 4u^4v^3 + \ u^4v^4$

$+ \sum_{n=8}^{\infty} \sum_{i=2}^{\infty} |D_{ve}\ (C_{n-3,\, i-1}\ )|\ u^{n-3}\, v^{i-1})$

$+ \ u^4v\ (uv + 2u^2v \ + u^2v^2 + 3u^3v + 3u^3v^2$

$+ u^3v^3 + 4u^4v + \ 6u^4v^2 + 4u^4v^3 + \ u^4v^4$

$+ \sum_{n=9}^{\infty} \sum_{i=2}^{\infty} |D_{ve}\ (C_{n-4,\, i-1}\ )|\ \ u^{n-4}\, v^{i-1})$

$f(u,v) - uv\ f(u,v) - u^2v\ f(u, v) - u^3v\ f(u, v) - u^4v\ f(u,v)$

$= u^5v^2\ (4 + 6v + 4v^2 + v^3) + u^5v^2$

$(3 + 3v + v^2 + 4u + 6uv + 4uv^2 + uv^3)$

$+ u^5v^2\ (2 + v + 3u + 3uv + uv^2 + 4u^2 + 6u^2v$

$+ 4u^2v^2 + u^2v^3)$

$+ u^5v\ (1 + 2u + uv + 3u^2 + 3u^2v + u^2v^2 + 4u^3 + 6u^3v$

$+ 4u^3v^2 + u^3v^3)$

$= u^5v^2\ (4 + 6u + 4v^2 + v^3 + 3 + 3v + v^2 + 4u + 6uv$

$+ 4uv^2 + 4v^3 + 2 + v + 3u + 3uv + uv^2 + 4u^2$

$+ 6u^2v + 4u^2v^2 + u^2v^3 + 1 + 2u + uv + 3u^2$

$+ 3u^2v + u^2v^2 + 4u^3 + 6u^3v + 4u^3v^2 + u^3v^3)$

$= u^5v^2\ (\ 10 + 10v + 5v^2 + v^3 + 9u + 10uv + 5uv^2$

$+ uv^2 + 7u^2 + 9u^2v + 5u^2v^2 + u^2v^3 + 4u^3$

$+ 6u^3v + 4u^3v^2 + u^3v^3)$

$$f(u, v) = \frac{u^5v^2(\ 10 + 10v + 5v^2 + 9u + 10uv + 5uv^2 + u^3 + uv^3 + 7u^2 + 9u^2v + 5u^2v^2 + u^2v^3 + 4u^3 + 6u^3v + 4u^3v^2 + u^3v^3)}{1 - uv - u^2v - u^3v - u^4v}$$

Conclusion

In this paper we obtain the vertex-edge dominating sets and vertex-edge domination polynomial of cycles. Similarly we can find vertex-edge domination sets and vertex-edge domination polynomial of specified graph.

4. References

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