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An Overview: Fractional Calculus Operators with Function

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Abstract
In this research, an overview of the historical development of fractional calculus is presented. Some basic definitions of fractional integration and differentiation are given with examples of some functions. The important analytical relations are introduced to be used later in the thesis. Introduction to fractional differential equations with its important applications in engineering and technologies, and numerical treatment for the solution of differential equation of fractional order are also provided.

Keywords: Function, Fractional, Calculus, Derivation, Integral, Equation

Introduction and Basic Definitions
In this section, basic theory of some special functions and definitions of fractional integral and derivative is given, which will be used in the next chapters. The fractional integral and derivative have been expressed in the literature in a variety of ways, including Riemann-Liouville, Caputo, Erdélyi-Kober, Hadamard, Grünwald-Letnikov and Riesz type etc. Equivalence of these definitions for some function has been given in standard fractional calculus reference books [7]. All these definitions have their own importance and advantages in different types of mathematical problems. Riemann-Liouville and Caputo definitions of fractional derivative are used in our study. Firstly we introduce the Gamma and Mittag-leffler functions, and then other definitions are presented which play significant role to develop a theory for differential of arbitrary order, as well as, in the theory of FDE equations.

Gamma Function
Probably, one of the fundamental functions of the fractional calculus is Euler's Gamma function, which is basically providing the generalization of factorial, *n!* And permits *n* and permits to take as real and even complex values.
The definition of the gamma function is given by the integral:

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx \qquad \text{.........................2.1}$$

The gamma function can also be defined by the limit representation as:

$$\Gamma(t) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad \mathrm{Re}(x) > 0 \qquad \text{..........2.2}$$

Two basic properties of gamma function are given as

$$\Gamma(t+1) = t\Gamma(t) = n(n-1)! = n! \qquad \text{..............................2.3}$$

And the second important property is that it has simple pole at the points ***x=-n,(n=0,1,2…,)***

Mittag-Leffler Function
G. M. Mittag-Leffler has developed a function, for one-parameter generalization of exponential function, which is represented by [2].

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha x+1)}. \qquad \text{.........................3.1}$$

It was introduced in [3] and also studied by A. Wiman Mittag-leffler function (MLF) of two parameters was in fact developed by Agarwal using series expansion and is defined as

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha x + \beta)}, \quad (\alpha > 0, \beta > 0)$$

…3.2

For ***β=1*** he above relation is reduced to MLF function in one parameter. It can be obtained from the definition (3.2) that,

$$E_{1,1}(x) = e^x, \quad E_{1,2}(x) = \frac{e^x - 1}{x}, \quad E_{1,3}(x) = \frac{e^x - 1 - x}{x^2}$$

….3.3

The particular cases of MLF can be represented by hyperbolic sine and cosine functions

$$E_{2,1}(x^2) = \cosh(x), \quad E_{2,2}(x^2) = \frac{\sinh(x)}{x}$$

…………3.4

Similarly the relation of MLF with error function is given as

$$E_{1/2,1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\frac{n}{2}+1)} = e^{x^2} erfc(-x)$$

…..…3.5

Where the error function is defined by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2} dx.$$

…………………………….3.6

Another important particular case of the MLF used extensively in solving fractional order systems is given as:

$$\varepsilon_x(\nu, a) = t^{\nu} \sum_{n=0}^{\infty} \frac{(ax)^n}{\Gamma(\nu + n + 1)} = t^{\nu} E_{1,\nu+1}(ax)$$

…….3.7

Where ***v*** is representing a fraction and ***a*** is ***a*** constant.

Riemann Liouville Fractional Integral And Derivative

The definition of Riemann-Liouville fractional integral of order is given as ***v<0***

$$(I^{\nu} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau,$$

………...4.1

$$(I^0 f)(t) = f(t)$$

…………..4.2

Where is I^V the order fractional integral. Accordingly, the fractional derivative of order ***v>0*** is normally given as:

$$(D^{\nu} f)(t) = (\frac{d}{dt})^n (I^{n-\nu} f)(t) \quad (n - 1 < \nu \le n)$$

…………4.3

Where D^V is the fractional derivative and ***n*** is an integer. One of the important properties associated with it is that the Riemann Liouville fractional derivative is an inverse of the integral of the same order [5]. Let us apply the definition of Riemann Liouville fractional derivative to power function given as:

$$f(t) = (t-a)^{\alpha}$$

……………..4.4

Where ***α*** is an arbitrary real number and ***a*** is some constant. It can be evaluated and provided as [30].

$$D^{\nu}(t-a)^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+\nu)} (t-a)^{\alpha-\nu}$$

……..…4.5

Similarly, in case of exponential function $e^{\lambda t}$, it can be evaluated as,

$$D^{\nu}(e^{\lambda t}) = t^{-\nu} E_{1,1-\nu}(\lambda)$$

………………………………4.6

Caputo Fractional Derivative

The definition of derivative provided by Riemann-Liouville has certain limitations when it is used for modeling of real-world phenomena associated with fractional differential equations. Therefore, we introduce a modified definition of fractional differential operator $\boldsymbol{D^V}$ given by Caputo [6].

$$(D^{\nu} f)(t) = I^{n-\nu} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau \quad (n-1 < \nu \le n)$$

…..5.1

Where $\boldsymbol{I}^{V}$ is given in (5.2). The usual property of the Caputo integral operator is:

$$(I^{\nu} D^{\nu} f)(t) = f(t) - \sum_{k=0}^{n-1} f^{k}(0) \frac{t^{k}}{k!} \quad (n-1 < \nu \le n)$$

…………5.2

The Caputo fractional derivative is obtained by computing an ordinary derivative followed by the fractional integral, whereas the Riemann-Liouville is obtained in the reverse order. The use of Caputo fractional derivative allows the traditional homogeneous, as well as, inhomogeneous initial and boundary conditions occurring often in general application. However, both the formulations, Riemann Liouville and Caputo, coincide for homogeneous initial conditions.

Laplace transform operator is used to understand the difference between Riemann-Liouville and Caputo derivatives in terms of initial conditions accompanied by fractional differential equations. The formula for Laplace transform of Riemann-Liouville fractional derivative contains the initial condition with limiting values for its fractional derivative. Problems with such type of initial conditions are solved successfully [7], however its solution is useless, because there is no physical interpretation for such initial conditions. On the contrary, such problems do not arise by taking Laplace transform of Caputo fractional derivative.

References

1. C. F. M. Coimbra. Mechanics with variable-order differential operators.Annalen der Physik , 12(1112):692–703, 2003.
2. M. Davison and C. Essex. Fractional differential equations and initial value problems. The mathematical Scientist ,23(2):108–116, 1998.
3. K. Diethelm.The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer, Heidelberg, 2010.
4. G. S. Frederico and D. F. Torres. Fractional noether's theorem in the riesz–caputo sense.Applied Mathematics and Computation, 217(3):1023 – 1033, 2010.
5. K. Furati. A cauchy-type problem with a sequential fractional derivative in the space of continuous functions. Boundary Value Problems, 2012(1):58, 2012.
6. S. Gaboury, R. Tremblay, and B.-J. Fugere. Some relations involving a generalized fractional derivative operator. Journal of Inequalities and Applications, 2013:167, 2013.
7. R. Herrmann. Fractional Calculus: An Introduction for Physicists. World Scientific, River Edge, New Jerzey, 2 editions, 2014.
8. U. N. Katugampola. New approach to a generalized fractional integral. Applied Mathematics and Computation,218(3):860–865, 2011.
9. G. B. Loghmani and S. Javanmardi. Numerical methods for sequential fractional differential equations for caputo operator. Bull. Malays. Math. Sci. Soc., 35(2):315–323, 2012.
10. K. S. Miller and B. Ross.An introduction to the fractional calculus and fractional differential equations . Wiley, New York, 1993.