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Oscillation of Third-Order Neutral Difference Equations

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ABSTRACT

In this paper, we are concerned with the oscillation of third order nonlinear neutral delay difference equations and some sufficient conditions for oscillation of solutions of third nonlinear neutral delay difference equations of the type

$$\Delta(a(n)(\Delta^2(x(n)+p(n)x(\sigma(n))))^{\gamma})+q(n)x^{\gamma}(\tau(n))=0$$

are obtained . An example is provided to illustrate the main results.

INTRODUCTION

We consider the nonlinear neutral delay difference equation of the form

$$\Delta(a(n)(\Delta^2(x(n)+p(n)x(\sigma(n))))^{\gamma})+q(n)x^{\gamma}(\tau(n))=0 \qquad (1)$$

where γ is a quotient of odd positive integers and $n\in N(n_0)=\{n_0,n_0+1,...\}$, n_0 is a nonnegative integer, subject to the following conditions.

$(H)a(n),p(n),q(n)$ are positive sequence . $0\le p(n)\le p<1,\tau$ and σ are positive integers and *a(n)* satisfies $\sum_{s=n_0}^{\infty}\frac{1}{a(s)^{\frac{1}{\gamma}}}=\infty$.

In recent years, the oscillation theory and asymptotic behaviour of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have appeared [1]. Determining oscillation criteria for particular second order difference equations has received a great deal of attention in the last few years. Compared to second order difference equations, the study of oscillation and asymptotic behaviour of third order difference equations has received considerably less attention [7 – 9].

Let $\theta=\max\{\sigma(n),\tau(n)\}$. By a solution of equation (1) we mean a real sequence *x(n)* is defined for all $n\ge n_0-\theta$ satisfies (1) for all $n\ge n_0$. A nontrivial solution *x(n)* is said to be oscillatory if it is neither eventually positive or eventually negative; otherwise , it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2. MAIN RESULTS

Lemma 2.1. Let *x(n)* be a positive solution of (1). Then there are only the following two case for $z(n)=x(n)+p(n)x(\sigma(n))$

$$z(n)>0, \Delta z(n)>0, \Delta^2 z(n)>o,$$

$$z(n)>0, \Delta z(n)<0, \Delta^2 z(n)>o,$$ where n is sufficiently large.

Proof. Assume that *x(n)* is a positive solution of (1) on $n \geq n_0$. We see that $z(n)>x(n)>0$ and

$$\Delta(a(n)(\Delta^2(z(n)))^\gamma)=-q(n)x^\gamma(\tau(n)) \qquad (2)$$

$\Delta(a(n)(\Delta^2(z(n)))^\gamma)$ is decreasing and of one sign. Therefore $\Delta^2 z(n)$ is also of one sign. We have two possibilities $\Delta^2 z(n)<0$ or $\Delta^2 z(n)>0$ for $n \geq n_1$ by (2). If we choose $\Delta^2 z(n)<0$, then there exits a constant $M>0$ such that

$a(n)(\Delta^2(z(n)))^\gamma \leq -M<0$

Summing the above inequality from n_1 and $n-1$, we obtain $\Delta z(n) \leq \Delta z(n_1) - M^{\frac{1}{\gamma}} \sum_{t=s}^{n-1} \frac{1}{a(s)^{\frac{1}{\gamma}}}$

Letting $n \to \infty$ and using (H), we obtain $\Delta z(n) \to -\infty$. Thus $\Delta^2 z(n)<0$ eventually. But $\Delta^2 z(n)<0$ and $\Delta z(n)<0$. Hence $z(n)<0$ for $n \geq n_1$, which is a contradiction. This contradiction proves $\Delta^2 z(n)>o,$ and we have only two case (i),(ii) for *z(n)*. The proof is complete.

Lemma 2.2. Let *x(n)* be a positive solution of equation (1) and the corresponding *z(n)* satisfies (ii). If

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a(s)} \sum_{t=s}^{\infty} q(t) \right)^\gamma = \infty, \qquad (3)$$

Then $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} z(n) = 0$.

Proof. Suppose that *x(n)* is a positive solution of (1). Since $z(n)>0$ and $\Delta z(n)>0$, then there exists a finite limit, $\lim_{n\to\infty} z(n)=\ell$. We shall prove that $\ell=0$. Assume that $\ell>0$. Then for any $\in>0$. We have $l+\in>z(n)>$ eventually .choose $0<\in<\frac{1(1-p)}{p}$. It is easy to verify that

$$x(n)=z(n)-p(n)x(\sigma(n))>\ell - pz(\sigma(n))>\ell - p(\ell+\in)=k(\ell+\in)>kz(n)$$

where $k=\frac{\ell-p(\ell+\in)}{l+\in}>0$. Using the above inequality , we obtain from (2)

$$\Delta(a(n)(\Delta^2(z(n))^\gamma)) \leq -q(n)k^\gamma z^\gamma(\tau(n)).$$

Summing the above inequality from *n* to ∞

$$a(n)(\Delta^2(z(n)))^\gamma \geq k^\gamma \sum_{s=n}^{\infty} q(s) z^\gamma(\tau(s))$$

Using $z^\gamma(\tau(n)) \geq \ell^\gamma$, we get

$$\Delta^2 z(n) \geq k\ell \left(\frac{1}{a(n)} \sum_{s=n}^{\infty} q(s) \right)^{\frac{1}{\gamma}}$$

Summing again from n to ∞, we have

$$-\Delta z(n) \geq k\ell \sum_{s=n}^{\infty} \left(\frac{1}{a(s)} \sum_{t=s}^{\infty} q(t) \right)^{\frac{1}{\gamma}}$$

Again taking summing n to ∞,

$$z(n_1) \geq k\ell \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a(i)} \sum_{s=n}^{\infty} q(t) \right)^{\frac{1}{\gamma}}$$

This contradicts (2). Then $\ell = 0$. Moreover the inequality $0 \leq x(n) \leq z(n)$ implies that $\lim_{n\to\infty} x(n) = 0$ and the proof is complete .

Lemma 2.3. Assume that $u(n) > 0, \Delta u(n) \geq 0, \Delta^2 u(n) \leq 0$ for all $n \geq n_0$. Then for each $\ell \in (0,1)$ there exists an integer $N \geq n_0$ such that $\frac{u(\tau(n))}{\tau(n)} \leq \ell \frac{u(n)}{n}$ for $n \geq N$.

Proof. From the monotonicity property of $\Delta u(n)$, we have

$$u(n) - u(\tau(n)) = \sum_{s=\tau(n)}^{n-1} \Delta u(s) \leq \Delta u(\tau(n))(n - \tau(n))$$

$$\frac{u(n)}{u(\tau(n))} \leq 1 + \frac{u(n)}{u(\tau(n))}(n - \tau(n)) \tag{4}$$

Also

$$u(\tau(n)) \geq u(\tau(n)) - u(n_0) \geq \Delta u(\tau(n))(\tau(n) - n_0)$$

So, for each $l \in (0,1)$ and $N \geq n_0$ such that

$$\frac{u(n)}{\Delta u(\tau(n))} \geq \ell\tau(n), n \geq N . \tag{5}$$

Combining (4) and (5), we get

$$\frac{u(n)}{u(\tau(n))} \leq 1 + \frac{1}{\ell\tau(n)}(n - \tau(n))$$

$$\leq \left(1 - \frac{1}{\ell}\right) + \left(\frac{n}{\ell(\tau(n))}\right)$$

$$\frac{u(n)}{u(\tau(n))} \leq \left(\frac{n}{\ell(\tau(n))}\right)$$

and the proof is complete.

Lemma 2.4. Assume that $z(n)>0, \Delta z(n)>0, \Delta^2 z(n)>0, \Delta^3 z(n)\le 0$ for all $n\ge N$. Then $\frac{z(n+1)}{\Delta z(n)}\ge\frac{n-N}{2}$ for $n\ge N$.

Proof. From the monotonicity property of $\Delta^2 z(n)$, we have

$$\Delta z(n)=\Delta z(N)+\sum_{s=N}^{n-1}\Delta^2 z(s)\ge(n-N)\Delta^2 z(n)$$

Summing from *N* to *n-1*, we obtain

$$z(n)-z(N)\ge\sum_{s=N}^{n-1}(s-N)\Delta^2 z(s)$$

$$z(n+1)\ge z(n)\ge\Delta z(N)+(n-N)\Delta z(n)-z(n+1)+z(N)$$

$$z(n+1)\ge\frac{1}{2}(n-N)\Delta z(n)$$

Lemma 2.5. Assume that $\Delta z(n)>0, \Delta^2 z(n)>0, \Delta^3 z(n)<0$ for all $n\ge N$. Then $(n-N)\frac{\Delta^2 z(n)}{\Delta z(n)}\le 1$ for $n\ge N$.

Proof. The result follows from the following inequality

$$\Delta z(n)\ge\sum_{s=N}^{n-1}\Delta^2 z(s)\ge\Delta^2 z(s)(n-N)$$

Now, we present the oscillation results. For simplicity, we introduce the following notation

$$p_*=\liminf_{n\to\infty}\frac{n^\gamma}{a(n)}\sum_{s=n}^{\infty}p_l(s)\quad q_*=\limsup_{n\to\infty}\frac{1}{n}\sum_{s=n_0}^{\infty}\frac{s^{\gamma+1}}{a(s)}p_l(s)\tag{6}$$

where $p_\ell(s)=l^\gamma(1-p)^\gamma q(s)\left(\frac{\tau(s)}{s}\right)^\gamma\left(\frac{\tau(s)-N}{2}\right)^\gamma$ with $l\in(0,1)$ arbitrarily chosen and *N* large enough. Moreover for *z(n)* satisfying case (i), we define

$$w(n)=a(n)\left(\frac{\Delta^2 z(n)}{\Delta z(n)}\right)^\gamma\tag{7}$$

and

$$r=\liminf_{n\to\infty}\frac{n^\gamma w(n+1)}{a(n+1)}\text{ and }R=\limsup_{n\to\infty}\frac{n^\gamma w(n)}{a(n)}\tag{8}$$

Lemma 2.6. Assume that *a(n)* is non decreasing. Let *x(n)* be a positive solution of equation (1) (i) Let $p_*<\infty$ and $q_*<\infty$. Suppose that the corresponding *z(n)* satisfies case (i) of Lemma

(2.1). Then

$$p_* \le r - r^{1+\frac{1}{\gamma}} \text{ and } p_* + q_* \le 1 \qquad (9)$$

(ii) If $p_* = \infty$ or $q_* = \infty$, then *z(n)* does not belong under case (i) of Lemma (2.1).

Proof. Part (i): Assume that *x(n)* is a positive solution of equation (1) and the corresponding *z(n)* satisfies (i). First note that

$$x(n) = z(n) - p(n)x(\sigma(n)) > z(n) - p(n)z(\sigma(n)) \ge (1-p)z(n)$$

Using the above inequality in equation (1), we obtain

$$\Delta(a(n))\left(\Delta^2(z(n))^\gamma\right) \le -(1-p)^\gamma q(n) z^\gamma(\tau(n)) \le 0 \qquad (10)$$

The last inequality together with $\Delta(a(n)) \ge 0$ gives $\Delta^3 z(n) \le 0$. So there exists an integer $N \ge n_0$ such that *z(n)* satisfies $z(\tau(n)) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \le 0$ for $n \ge N$. From definition of *w(n)* and (10), we see that *w(n)* is positive and satisfies,

$$\Delta w(n) = \frac{\Delta\left(a(n)\left(\Delta^2(z(n))^\gamma\right)\right)}{(\Delta z(n))^\gamma} - \frac{a(n+1)\left(\Delta^2 z(n+1)\right)^\gamma \Delta(\Delta z(n))^\gamma}{(\Delta z(n))^\gamma (\Delta z(n+1))^\gamma}$$

$$\Delta w(n) \le \frac{-(1-p)^\gamma q(n) z^\gamma(\tau(n))}{(\Delta z(n))^\gamma} - \frac{\gamma}{a^{\frac{1}{\gamma}}(n+1)^\gamma} w^{1+\frac{1}{\gamma}}(n+1) \qquad (11)$$

From Lemma (2.3) with $u(n) = \Delta z(n)$, we have for ℓ the same as in $p_\ell(s)$

$$\frac{1}{\Delta z(n)} \ge \frac{\ell\tau(n)}{n} \frac{1}{\Delta z(\tau(n))}, \quad n \ge N$$

which with (11) gives

$$\Delta w(n) \le -\ell^\gamma q(n) \left(\frac{\tau(n)}{n}\right)^\gamma \left(\frac{z(\tau(n))}{\Delta z(\tau(n))}\right)^\gamma (1-p)^\gamma - \frac{\gamma}{a^{\frac{1}{\gamma}}(n+1)^\gamma} w^{1+\frac{1}{\gamma}}(n+1)$$

Using the fact from Lemma (2.4) that $z(n+1) \ge \frac{(n-N)}{2} \Delta z(n)$ we have,

$$\Delta w(n) + P_l(n) + \frac{\gamma}{a^{\frac{1}{\gamma}}(n+1)^\gamma} w^{1+\frac{1}{\gamma}}(n+1) \le 0 \qquad (12)$$

Since $P_\ell(n) > 0$ and $w(n) > 0$ for $n \ge N$, $\Delta w(n) \le 0$ and

$$\frac{-\Delta w(n)}{\gamma w^{1+\frac{1}{\gamma}}(n+1)} \ge \frac{1}{a^{\frac{1}{\gamma}}(n+1)} \quad \text{for } n \ge N$$

Summing the last inequality *N* to *n-1* and using the fact that w(n) is decreasing, we obtain

$$\frac{-w(n)+w(N)}{\gamma w^{1+\frac{1}{\gamma}}(n)} \geq \sum_{s=N}^{n-1} \frac{1}{a^{\frac{1}{\gamma}}(n+1)}$$

$$w(n) < \left(\frac{w(N)}{\gamma \sum_{s=N}^{n-1} \frac{1}{a^{\frac{1}{\gamma}}(n+1)}} \right)^{1+\frac{1}{\gamma}} \qquad (13)$$

which view of (H) implies that, $\lim_{n\to\infty} w(n) = 0$. On the other hand, from the definition of *w(n)* and Lemma(2.5), we see that

$$0 \leq r \leq R \leq 1 \qquad (14)$$

Now, we prove that the first inequality in (9) holds. Let $\in > 0$. Then from the definition of p_* and r we can choose an integer $n_2 \geq N$ sufficiently large that $\frac{n^\gamma}{a(n)} \sum_{s=n}^{\infty} p_l(s) \geq p_* - \in$ and $\frac{n^\gamma}{a(n+1)} w(n+1) \geq 1 - \in$ for $n \geq n_2$. Summing (12) from n to ∞ and using $\lim_{n\to\infty} w(n) = 0$, we have

$$w(n) \geq \sum_{s=n}^{\infty} p_l(s) + \gamma \sum_{s=n}^{\infty} \frac{w^{1+\frac{1}{\gamma}}(s+1)}{a^{\frac{1}{\gamma}}(s+1)}, n \geq n_2 \qquad (15)$$

Using the fact $\Delta a(n) \geq 0$, it follows from (15) that

$$\frac{n^\gamma w(n)}{a(n)} \geq p_* - \in + \frac{\gamma n^\gamma}{a(n)} \sum_{s=n}^{\infty} \frac{s^{\gamma+1} a(s+1) w^{1+\frac{1}{\gamma}}(s+1)}{s^{\gamma+1} a^{\frac{1}{\gamma}+1}(s+1)}$$

$$\geq (p_* - \in) + \frac{n^\gamma (r-\in)^{1+\frac{1}{\gamma}}}{a(n)} \sum_{s=n}^{\infty} \frac{\gamma a(s+1)}{s^{\gamma+1} a^{\frac{1}{\gamma}+1}(s+1)}$$

$$\geq (p_* - \in) + \frac{n^\gamma (r-\in)^{1+\frac{1}{\gamma}}}{a(n)} \sum_{s=n}^{\infty} \frac{\gamma a(s+1)}{s^{\gamma+1}}$$

$$\geq (p_* - \in) + n^\gamma (r-\in)^{1+\frac{1}{\gamma}} \sum_{s=n}^{\infty} \frac{\gamma}{s^{\gamma+1}}$$

and so

$$\frac{n^\gamma w(n)}{a(n)} \geq (p_* - \in) + n^\gamma (r-\in)^{1+\frac{1}{\gamma}} \sum_{s=n}^{\infty} \frac{\gamma}{s^{\gamma+1}} \qquad (16)$$

From (2.16) and $\sum_{s=n}^{\infty} \frac{\gamma}{s^{\gamma+1}} \geq \gamma \int_{n}^{\infty} \frac{ds}{s^{\gamma+1}}$, we have

$$\frac{n^\gamma w(n)}{a(n)} \geq (p_* - \in) + (r-\in)^{1+\frac{1}{\gamma}}$$

Taking *lim inf* of both sides as $n \to \infty$, we get

$$r \geq (p_* - \epsilon) + (r - \epsilon)^{1+\frac{1}{\gamma}}$$

Since $\epsilon > 0$ is arbitrary we get the result,

$$p_* \leq r - r^{1+\frac{1}{\gamma}} \qquad (17)$$

To complete the proof of part (I), it remains to prove the second inequality in (9). Multiplying the inequality (12) by $\frac{n^{\gamma+1}}{a(n)}$ and summing from n_2 to $n-1$, we obtain

$$\frac{n^{\gamma+1}}{a(n)} \sum_{s=n_2}^{\infty} \frac{s^{\gamma+1}\Delta w(n)}{a(s)} \leq -\sum_{s=n_2}^{\infty} \frac{s^{\gamma+1}}{a(s)} p_l(s) - \gamma \sum_{s=n_2}^{\infty} \left(\frac{s^{\gamma} w(s+1)}{a(s+1)} \right)^{\frac{\gamma+1}{\gamma}} \qquad (18)$$

By summation by parts, we obtain

$$\frac{n^{\gamma+1}}{a(n)} w(n) \leq \frac{n_2^{\gamma+1}}{a(n_2)} w(n_2) - \sum_{s=n_2}^{n-1} \frac{s^{\gamma+1}}{a(s)} P_l(s) - \gamma \sum_{s=n_2}^{n-1} \left(\frac{s^{\gamma} w(n_2)}{a(s+1)} \right)^{\frac{\gamma+1}{\gamma}} + \sum_{s=n_2}^{n-1} w(s+1) \Delta \left(\frac{s^{\gamma+1}}{a(s)} \right)$$

Since $\Delta a(n) \geq 0$, we have

$$\Delta \left(\frac{s^{\gamma+1}}{a(n)} \right) = \frac{a(s)\Delta(s^{\gamma+1}) - s^{\gamma+1}\Delta a(s)}{a(s)a(s+1)} \leq \frac{(\gamma+1)(s+1)^{\gamma}}{a(s+1)}$$

Hence,

$$\frac{n^{\gamma+1} w(n)}{a(n)} \leq \frac{n_2^{\gamma+1} w(n_2)}{a(n_2)} - \sum_{s=n_2}^{n-1} \frac{s^{\gamma+1} P_l(s)}{a(s)} - \sum_{s=n_2}^{n-1} \left[\frac{(\gamma+1)(s+1)^{\gamma} w(s+1)}{a(s+1)} - \gamma \left(\frac{s^{\gamma} w(s+1)}{a(s+1)} \right)^{\frac{\gamma+1}{\gamma}} \right]$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}$$

With $u = \frac{s^{\gamma} w(s+1)}{a(s+1)} > 0, A = \gamma$ and $B = (\gamma+1)\left(\frac{(s+1)}{s} \right)^{\gamma}$, we get

$$\frac{n^{\gamma+1} w(n)}{a(n)} \leq \frac{n_2^{\gamma+1} w(n_2)}{a(n_2)} - \sum_{s=n_2}^{n-1} \frac{s^{\gamma+1} P_l(s)}{a(s)} + \sum_{s=n_2}^{n-1} \left(\frac{s+1}{s} \right)^{\gamma(\gamma+1)}$$

It follows that

$$\frac{n^{\gamma} w(n)}{a(n)} \leq \frac{n_2^{\gamma+1} w(n_2)}{a(n_2)} - \frac{1}{n} \sum_{s=n_2}^{n-1} \frac{s^{\gamma+1} P_l(s)}{a(s)} + \frac{1}{n} \sum_{s=n_2}^{n-1} \left(\frac{s+1}{s} \right)^{\gamma(\gamma+1)} \qquad (19)$$

Taking the $\limsup$ of $n \to \infty$

$$R \leq -q_* + 1$$

Combining this with the inequalities in (17) and (14), we have

$$p_* \leq r - r^{1+\frac{1}{\gamma}} \leq r \leq R \leq -q_* + 1$$

which gives the desired second inequality in (9). The proof of part (I) is complete.

Part (ii): Assume *x(n)* positive solution of (1). We shall show that *z(n)* does not belong to case (i) of Lemma (2.1). Assume the contrary. First assume $p_* = \infty$. This is exactly as in the first part, we obtain (15). Then

$$\frac{n^\gamma}{a(n)} w(n) \geq \frac{n^\gamma}{a(n)} \sum_{s=n}^{\infty} P_l(s)$$

Taking $\liminf$ of both sides at $n \to \infty$, we obtain in view of (1)

$1 \geq r \geq \infty$

This is a contradiction. Next we assume that $q_* = \infty$. Then taking $\liminf$ and $\limsup$ on the left and right sides of (19) respectively, we obtain

$0 \leq R \leq -\infty$

This contradiction completes the proof.

Now we are ready to present the following oscillation criterion for equation (1).

Theorem 2.7. Assume that condition (3) holds and *a(n)* is nondecreasing. Let *x(n)* be a solution of (1). If

$$p_* = \liminf_{n \to \infty} \frac{n^\gamma}{a(n)} \sum_{s=n}^{\infty} P_l(s) > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \qquad (20)$$

then *x(n)* is oscillatory or $x(n) \to 0$ as $n \to \infty$.

Proof. Let *x(n)* be a nonoscillatory solution of equation (1) without loss of generality we may assume that *x(n)* is a positive solution of equation (1). If $p_* = \infty$, then by Lemma (2.6), *z(n)* does not belong to case (i) of Lemma (2.1). That is, *z(n)* has to satisfy (ii), from Lemma (2.2), we see that $\liminf_{n \to \infty} x(n) = 0$.

Next, we assume that $p_* < \infty$. We shall discuss two possibilities. If for *z(n)* case (ii) holds, then exactly as above we are led, by Lemma (2.6), to $\liminf_{n \to \infty} x(n) = 0$.

Now we assume that for *z(n)* case (i) holds. Let w(n) and r be defined by (7) and (8) respectively, then from Lemma (2.6) we see that *r* satisfies the inequality

$p_* \leq r - r^{1+\frac{1}{\gamma}}$

Using the inequality $Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$ with A = B = 1 and u = 1, we obtain that

$p_* \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}$

which contradicts (20). This completes the proof.

Corollary 2.8. Assume that condition (3) holds and *a(n)* is nondecreasing. Let *x(n)* be a solution of (1). If

$$\liminf_{n\to\infty}\frac{n^{\gamma}}{a(n)}\sum_{s=n}^{\infty}q(s)\frac{\tau^{2\gamma}(s)}{s^{\gamma}}>\frac{(2\gamma)^{\gamma}}{(\gamma+1)^{\gamma+1}(1-p)^{\gamma}} \qquad (21)$$

then *x(n)* is oscillatory or $x(n)\to 0$ as $n\to\infty$.

Proof. We shall show that condition (21) implies condition (20). First note that for any $\ell\in(0,1)$ there exists an integer n_1 such that $\tau(n)-N\le\ell(n), n\ge n_1$. Therefore,

$$P_l(n)\ge\frac{\ell^{2\gamma}(1-p)^{\gamma}}{2\gamma}\frac{\tau^{2\gamma}(n)}{n^{\gamma}}q(n), n\ge n_1 \qquad (22)$$

On the other hand (21) implies that for some $l\in(0,1)$

$$\liminf_{n\to\infty}\frac{n^{\gamma}}{a(n)}\sum_{s=n}^{\infty}q(s)\frac{\tau^{2\gamma}(s)}{s^{\gamma}}>\frac{1}{\ell^{2\gamma}}\frac{(2\gamma)^{\gamma}}{(\gamma+1)^{\gamma+1}(1-p)^{\gamma}} \qquad (23)$$

Combining (22) with (23), we obtain (20).

Theorem 2.9. Assume that condition (3) holds and *a(n)* is nondecreasing. Let *x(n)* be a solution of (1). If

$$p_*+q_*>1 \qquad (24)$$

then *x(n)* is oscillatory or satisfies $\lim_{n\to\infty}=0$.

Proof. Let *x(n)* be a nonoscillatory solution of equation (1) without loss of generality we may assume that *x(n)* is a positive solution of equation (1). If $p_*=\infty$ or $q_*=\infty$, then by Lemma (2.6), *z(n)* does not belong to case (i) of Lemma (2.1). That is, *z(n)* has to satisfy (ii), from Lemma (2.2), we see that $\liminf_{n\to\infty}x(n)=0$.

Next, we assume that $p_*<\infty$ or $q_*<\infty$. We shall discuss two possibilities. If for *z(n)* case (ii) holds, then exactly as above we are led, by Lemma (2.2), to $\liminf_{n\to\infty}x(n)=0$. Now we assume that for *z(n)* case (i) holds. Let *w(n)* and r be defined by (7) and (8) respectively, then from Lemma (2.6) we see that p_* or q_* satisfies the inequality, which contradicts (24). This complete the proof.

As a consequence of Theorem (2.9), we have the following results.

Corollary 2.10. Assume that condition (3) holds and *a(n)* is nondecreasing. Let *x(n)* be a solution of (1). If

$$p_*=\liminf_{n\to\infty}\frac{1}{n}\sum_{s=n_0}^{n-1}\frac{s^{\gamma+1}}{a(s)}P_{\ell}(s)>1 \qquad (25)$$

then *x(n)* is oscillatory or satisfies $\lim_{n\to\infty}=0$.

As a matter of fact we can again slightly simplify function $P_{\ell}(n)$ in (25).

Corollary 2.11. Assume that condition (3) holds and *a(n)* is nondecreasing. Let *x(n)* be a solution of (1). If

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{s=n_0}^{n-1}\frac{s\tau^{2\gamma}(s)}{a(s)}q(s)\ge\frac{2\gamma}{(1-p)^{\gamma}} \qquad (26)$$

then *x(n)* is oscillatory or satisfies $\lim_{n\to\infty} = 0$.

The proof is similar to that of Corollary (2.8) and hence the details are omitted. We conclude this section with two examples.

Example 2.12. Consider the third order nonlinear difference equation

$$\Delta\left(n\left(\Delta^2\left(x(n)+\frac{1}{3}x(n-1)\right)\right)^3\right)+\left(\frac{8}{3}\right)^3(2n+1)(n-2)=0 \qquad (27)$$

It is easy to see that condition (3) holds. Hence, by Corollary (2.8), we see that every solution of equation (27) is either oscillatory or converges to zero as $n\to\infty$.

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